NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE

(Accredited by NAAC, Approved by AICTE New Delhi, Affiliated to APJKTU)

Pampady, Thiruvilwamala(PO), Thrissur(DT), Kerala 680 588

DEPARTMENT OF MECHATRONICS

COURSE MATERIALS

MAT 201- PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS

VISION OF THE INSTITUTION

To mould our youngsters into Millennium Leaders not only in Technological and Scientific Fields but also to nurture and strengthen the innate goodness and human nature in them, to equip them to face the future challenges in technological break troughs and information explosions and deliver the bounties of frontier knowledge for the benefit of humankind in general and the down-trodden and underprivileged in particular as envisaged by our great Prime Minister Pandit Jawaharlal Nehru

MISSION OF THE INSTITUTION

To build a strong Centre of Excellence in Learning and Research in Engineering and Frontier Technology, to facilitate students to learn and imbibe discipline, culture and spirituality, besides encouraging them to assimilate the latest technological knowhow and to render a helping hand to the under privileged, thereby acquiring happiness and imparting the same to others without any reservation whatsoever and to facilitate the College to emerge into a magnificent and mighty launching pad to turn out technological giants, dedicated research scientists and intellectual leaders of the society who could prepare the country for a quantum jump in all fields of Science and Technology

ABOUT DEPARTMENT

- Established in: 2013
- Course offered: B.Tech Mechatronics Engineering
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

To develop professionally ethical and socially responsible Mechatronics engineers to serve the humanity through quality professional education.

DEPARTMENT MISSION

1) The department is committed to impart the right blend of knowledge and quality education to create professionally ethical and socially responsible graduates.
2) The department is committed to impart the awareness to meet the current challenges in technology.
3) Establish state-of-the-art laboratories to promote practical knowledge of mechatronics to meet the needs of the society

PROGRAMME EDUCATIONAL OBJECTIVES

I. Graduates shall have the ability to work in multidisciplinary environment with good professional and commitment.
II. Graduates shall have the ability to solve the complex engineering problems by applying electrical, mechanical, electronics and computer knowledge and engage in life long learning in their profession.
III. Graduates shall have the ability to lead and contribute in a team entrusted with professional social and ethical responsibilities.
IV. Graduates shall have ability to acquire scientific and engineering fundamentals necessary for higher studies and research.

PROGRAM OUTCOME (PO'S)

Engineering Graduates will be able to:

PO 1. Engineering knowledge: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO 2. Problem analysis: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO 3. Design/development of solutions: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO 4. Conduct investigations of complex problems: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO 5. Modern tool usage: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO 6. The engineer and society: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO 7. Environment and sustainability: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO 8. Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO 9. Individual and team work: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO 10. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO 11. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO 12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOME(PSO'S)

PSO 1: Design and develop Mechatronics systems to solve the complex engineering problem by integrating electronics, mechanical and control systems.

PSO 2: Apply the engineering knowledge to conduct investigations of complex engineering problem related to instrumentation, control, automation, robotics and provide solutions.

COURSE OUTCOME

CO 1	Understand the concept and the solution of partial differential equation.
CO 2	Analyse and solve one dimensional wave equation and heat equation.
CO 3	Understand complex functions, its continuity differentiability with the use of Cauchy-
	Riemann equations.
CO 4	Evaluate complex integrals using Cauchy's integral theorem and Cauchy's
	integral formula, understand the series expansion of analytic function
CO 5	Understand the series expansion of complex function about a singularity and
	Apply residue theorem to compute several kinds of real integrals.

CO VS PO'S AND PSO'S MAPPING

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PS01	PSO1
CO 1	3	3	3	3	2	1	-	-	-	2	-	2	1	1
CO 2	3	3	3	3	2	1	-	-	-	2	-	2	1	1
CO 3	3	3	3	3	2	1	-	-	-	2	-	2	1	1
CO 4	3	3	3	3	2	1	-	-	-	2	-	2	1	1
CO 5	3	3	3	3	2	1	-	-	-	2	-	2	1	1

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

SYLLABUS

Module 1 (Partial Differential Equations) (8 hours)

(Text 1-Relevant portions of sections 17.1, 17.2, 17.3, 17.4, 17.5, 17.7, 18.1, 18.2)
Partial differential equations, Formation of partial differential equations –elimination of arbitrary constants-elimination of arbitrary functions, Solutions of a partial differential equations, Equations solvable by direct integration, Linear equations of the first order-Lagrange's linear equation, Non-linear equations of the first order -Charpit's method, Solution of equation by method of separation of variables.

Module 2 (Applications of Partial Differential Equations) (10 hours)
(Text 1-Relevant portions of sections 18.3,18.4, 18.5)

One dimensional wave equation- vibrations of a stretched string, derivation, solution of the wave equation using method of separation of variables, D'Alembert's solution of the wave equation, One dimensional heat equation, derivation, solution of the heat equation

Module 3 (Complex Variable – Differentiation) (9 hours)
(Text 2: Relevant portions of sections13.3, 13.4, 17.1, 17.2 , 17.4)

Complex function, limit, continuity, derivative, analytic functions, Cauchy-Riemann equations, harmonic functions, finding harmonic conjugate, Conformal mappings- mappings $W= Z^2$, $W=e^Z$ .Linear fractional transformation W = 1/Z fixed points, Transformation W=Sin Z

MATHEMATICS
(From sections 17.1, 17.2 and 17.4 only mappings $W= Z^2$, $W=e^Z$,W = 1/Z and W=Sin Z problems based on these transformation need to be discussed)

Module 4 (Complex Variable – Integration) (9 hours)
(Text 2- Relevant topics from sections14.1, 14.2, 14.3, 14.4,15.4**)**

Complex integration, Line integrals in the complex plane, Basic properties, First evaluation method-indefinite integration and substitution of limit, second evaluation method-use of a representation of a path, Contour integrals, Cauchy integral theorem (without proof) on simply connected domain,Cauchy integral theorem (without proof) on multiply connected domain Cauchy Integral formula (without proof), Cauchy Integral formula for derivatives of an analytic function, Taylor's series and Maclaurin series.

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Module 5 (Complex Variable – Residue Integration) (**9 hours**)
(Text 2- Relevant topics from sections16.1, 16.2, 16.3, 16.4 **)**

Laurent's series(without proof), zeros of analytic functions, singularities, poles, removable singularities, essential singularities, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral using residue theorem, Residue integration of real integrals – integrals of rational functions of Cos θ and Sinθ , integrals of improper integrals of the form $\int_{-\infty}^{\infty} f(x)dx$ with no poles on the real axis.($\int_{A}^{B} f(x)dx$ whose integrand become infinite at a point in the interval of integration is excluded from the syllabus),

Textbooks:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 44th Edition, 2018.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10th Edition, John Wiley & Sons, 2016.

References:

1. Peter V. O'Neil, Advanced Engineering Mathematics, Cengage, 7th Edition, 2012

QUESTION BANK

MODULE I

Q:NO:	QUESTIONS	CO	KL	PAGE NO:
1	Derive a partial differential equation from the relation $z = f(x + at) + g(x - at)$	CO1	K1	14
2	Derive a partial differential equation from the relation $z = yf(x) + xg(y)$	CO1	K3	15
3	Find the differential equation of all planes which are at a constant distance a from the origin	CO1	K1	15
4	Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + sin(2x - y) = 0$	CO1	K3	16
5	Use Charpit's methods to solve $q + xp = p^2$	CO1	K3	21
6	Use Charpit's methods to solve $(p^2 + q^2)y = qz$	CO1	K3	48
7	Solve $x(y - z)p + y(z - x)q = z(x - y)$	CO1	K2	30
8	Solve $(y - z)p + (x - y)q = z - x$	CO1	K2	32
9	Solve $xydx + y^2dy = zxy - 2x^2$	CO1	K2	32
10	Solve by the method of separation of variables $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$	CO1	K3	40
11	Using the method of separation of variables, solve $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = \mathbf{0}$	CO1	K3	38
12	Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x, 0) = 6\,e^{-3x}$	CO1	K3	44

MODULE II				
Q:NO:	**QUESTIONS**	**CO**	**KL**	**PAGE NO:**
1	Derive One dimensional wave equation	CO2	K2	50
2	Derive the solution of one dimensional wave equation	CO2	K2	50
3	A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 sin^3 \frac{\pi x}{l}$ Find the displacement of the string at any time.	CO2	K3	58
4	A tightly stretched string with fixed end points $x = 0\ and\ x = l$ is initially in a position given by $y = y_0 sin^3 \frac{\pi x}{l}$. If it is released from rest from this position find the displacement $y(x, t)$	CO2	K3	62
5	A transversely vibrating string of length 'a' is stretched between two points A and B. The initial displacement of each point of the string is zero and the initial velocity at a distance x from A is kx(a-x). Find the form of string at any subsequent time.	CO2	K3	62
6	Derive Solution of one dimensional wave equation using D Alembert's method	CO2	K1	63
7	Derive One dimensional heat equation	CO2	K1	64
8	Derive Solution of one dimensional heat equation using variable separable method	CO2	K2	66
9	Find the temperature $U(x, t)$ of a homogeneous bar of heat conducting material of length l whose end points are kept at zero	CO2	K3	67

	temperature and whose initial temperature is given by $\frac{ax(l-x)}{l^2}$			
10	A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is $u(x, 0) = f(x) = \begin{cases} x \quad , & 0 < x < 50 \\ 100 - x \, , & 50 < x < 100 \end{cases}$ Find the temperature (x, t) at any time.	CO2	K3	67

MODULE III

Q.NO:	QUESTIONS	CO	KL	PAGE NO:
1	Check whether the function $f(z) = \frac{Re(z^2)}{\lvert z \rvert}$ is continuous at $z = 0$ given $f(0) = 0$	CO3	K2	68
2	Check whether $f(z) = e^{-2x}\,(cos\ 2y - \ isin\ 2y)$ is analytic	CO3	K3	68
3	Show that $f(z) = e^z$ is analytic for all z. Find its derivative.	CO3	K1	69
4	Prove that the function f(x, y) = $x^3 - 3xy^2 - 5y$ is harmonic everywhere. Find its harmonic conjugate.	CO3	K3	70
5	If the $function\ \ u = ax^3 + bxy$ is harmonic then find a and b. Also find its harmonic conjugate.	CO3	K3	71
6	Verify $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function $v\ of\ u$	CO3	K3	71
7	$Show\ that\ f(z) = \lvert z \rvert^2$ is no where analytic	CO3	K1	72
8	Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function V and express $u + iv$ as an analytic function of z.	CO3	K3	73
9	Find the image of the regions 2<|z|<3 and |argz|<$\frac{\pi}{4}$ under the	CO3	K2	73

	transformation $w = z^2$ and plot it			
10	Find the fixed points of the bilinear transformation i) $w = \frac{z-1}{z+1}$	CO3	K1	77
11	Find the fixed points of the bilinear transformation w =$\frac{3z-4}{z-1}$	CO3	K1	78
12	Find the image of the following infinite strips under the mapping $w = \frac{1}{z}$	CO3	K2	79

MODULE IV

Q.NO	QUESTIONS	CO	KL	PAGE NO:
1	Evaluate $\int_0^{1+i} x^2 + iy \quad dz \qquad along\ the\ line\ y =$	CO4	K1	95
2	Evaluate $\int_c \frac{dz}{(z-1)(z-2)}$ where C is $\|z\| = \frac{1}{2}$	CO4	K2	95
3	Use Cauchy's Integral formula to evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2 \ dz}{(z-1)(z-3)}$	CO4	K3	92
4	Evaluate $\int_C \frac{3z-1}{z^3-z} dz\ where\ C\ is\ (i)\ \|z\| = \frac{1}{2} \quad and\ (ii)\ \|z\| = 2$	CO4	K1	96
5	Expand $\frac{1}{z+2}$ at z=1 as Taylor's series	CO4	K2	98

6	Expand f(z)=$\frac{1}{z^2+3z+2}$ in the region $1 < \|z\| < 2$	CO4	K3	96
7	Find the Maclaurine series expansion of f(z)= 1/(1-z) and state the radius of convergence	CO4	K3	96
8	Evaluate $\int_c Re(z)dz$ where C is a straight line from 0 to 1+2i.	CO4	K2	87
9	Integrate $\frac{z^2}{z^2-1}$ counter clockwise around the circle $\|z-1-i\| = \frac{\pi}{2}$ by Cauchy's integral formula.	CO4	K2	91
10	Evaluate $\int_c \frac{z+4 \ dz}{z^2+2z+5)}$ where C is i) $\|z+1-i\| = 2$ ii) $\|z+1+i\| = 2$	CO4	K3	95
11	Evaluate $\int_c \frac{sinz \ dz}{(z-\frac{\pi}{2})^2}$ where C is the circle $\|z\| = 2$	CO4	K3	96
12	Evaluate $\int_c \frac{dz}{(z-1)(z-2)}$ where C is $\|z\| = \frac{1}{2}$	CO4	K3	101
	MODULE V			
Q.NO	**QUESTIONS**	**CO**	**KL**	**PAGE NO:**
1	Determine & classify the singular points for the functions i) $\frac{x-sinz}{z^3}$ ii) $e^{-\frac{1}{z^2}}$	CO5	K1	120
2	Determine & classify the singular points for the functions i) $\frac{sinz}{(z-\pi)^2}$ ii) $tanz$	CO5	K1	115
3	Discuss the singularities of $f(z) = \frac{(z^2-1)(z-2)^2}{(\sin \pi z)^2}$	CO5	K2	127
4	Find the residues of $\frac{z+2}{(z-2)(z+1)^2}$	CO5	K3	115

5	Find the residues of $f(z) = \frac{1+e^z}{\sin z + z\cos z}$	CO5	K3	140
6	Evaluate $\int_c \frac{dz}{z^3(z-1)}$ $where\ \ c\ is\ \ |z| = 2$	CO5	K3	142
7	Evaluate $\int_c \frac{(\cos \pi z^2 + \sin \pi z^2)dz}{(z+1)(z+2)}$ $where\ |z| = 3$ using Residue theorem	CO5	K3	135
8	Find the Laurent series expansion of $\frac{z+4}{(z+3)(z-1)^2}$ in the following region (a) 0<|z-1|<4 (b) |z-1|>4	CO5	K2	113
9	Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ $where\ a > b > 0$	CO5	K3	154
10	Evaluate $\int_0^{\infty} \frac{dx}{x^2+1}$	CO5	K3	156
11	Using Residue theorem find $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$	CO5	K3	155
12	$Calculate \int_c \frac{e^z\,dz}{\cos\pi z}$,C : $|z| = 1$ using residue theorem	CO 5	K3	157

MODULE - I

PARTIAL DIFFERENTIAL EQUATIONS

The following notations are adopted through out the study of partial differential equation.

Let $z = f(x, y)$

$$\boxed{\frac{\partial z}{\partial x} = p \qquad \frac{\partial^2 z}{\partial y^2} = t \qquad \frac{\partial z}{\partial y} = q \qquad \frac{\partial^2 z}{\partial x \partial y} = s \qquad \frac{\partial^2 z}{\partial x^2} = r}$$

Elimination of arbitrary constants.

Q) Find PDE from $z = (x-a)^2 + (y-b)^2$

Solution:-

$$\frac{\partial z}{\partial x} = 2(x-a) = p$$

$$\Rightarrow (x-a) = \frac{p}{2}$$

$$\frac{\partial z}{\partial y} = 2(y-b) = q$$

$$\Rightarrow (y-b) = \frac{q}{2}$$

$$\therefore \quad z = (x-a)^2 + (y-b)^2$$

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$\Rightarrow 4z = p^2 + q^2$ which is the required PDE.

(2) Find PDE by eliminating arbitrary constants

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Soln:

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{--- (1)}$$

differentiating (1) partially w.r.to x,

$$2\frac{\partial z}{\partial x} = \frac{1}{a^2}2x + 0 = \frac{2x}{a^2}$$

$$\Rightarrow \cancel{2}\frac{\partial z}{\partial x} = \frac{\cancel{2}\cdot x}{a^2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{x}{a^2}$$

$$\Rightarrow \boxed{\frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x}}$$

differentiating (1) partially w.r.to y,

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{y}{b^2}$$

$$\Rightarrow \boxed{\frac{1}{b^2} = \frac{1}{y}\cdot\frac{\partial z}{\partial y}}$$

$\therefore$ (1) $\Rightarrow$ $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$\Rightarrow 2z = x^2\cdot\frac{1}{x}\frac{\partial z}{\partial x} + y^2\cdot\frac{1}{y}\frac{\partial z}{\partial y}$$

$$\Rightarrow 2z = x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y}$$

$\Rightarrow 2z = xp + yq$ which is the required PDE.

(3) Form PDE by eliminating arbitrary constants

$z = ax + by + a^2 + b^2$

Soln:

$$z = ax + by + a^2 + b^2 \quad \text{--- (1)}$$

differentiating (1) w.r.to x,

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

differentiating (1) w.r.to y,

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

$$\therefore z = ax + by + a^2 + b^2$$

$$z = px + qy + p^2 + q^2$$

(4) Find the differential equation of all spheres of fixed radius having their centres in xy plane

Soln:

Equation of spheres with centre $(h, k, 0)$ in xy plane and radius 'r' is

$$(x-h)^2 + (y-k)^2 + z^2 = r^2 \quad \text{--- (1)}$$

differentiating (1) partially w.r.to x,

$$2(x-h) + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x - h = -z\frac{\partial z}{\partial x}$$

$$\Rightarrow \boxed{x - h = -zp} \qquad \left[\because \frac{\partial z}{\partial x} = p\right.$$

Now differentiating (1) partially w.r.to y,

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad y-k = -z\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad \boxed{y-k = -zq} \qquad \because \frac{\partial z}{\partial y} = q$$

$\therefore$ eqn ① $\quad (x-h)^2+(y-k)^2+z^2=r^2$

$$\Rightarrow (-pz)^2 + (-qz)^2 + z^2 = r^2$$

$$\Rightarrow z^2\left[p^2+q^2+1\right] = r^2$$ which is the required PDE.

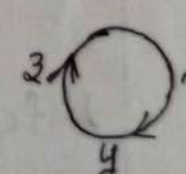
Eliminating Arbitrary functions

If the function is of the form $\phi(u,v)=0$ where u, v are arbitrary functions.

Then the PDE is $Pp + Qq = R$

where $P = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}$ $\qquad Q = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix}$

and $\quad R = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$



where $u_x, u_y\ u_z$ -- are derivatives.

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(1) Form PDE by eliminating arbitrary functions.

$xyz = \phi(x+y+z)$

Soln:

$$xyz = \phi(x+y+z)$$

$$\phi(u,v) = 0 \Rightarrow \phi(xyz,\ x+y+z) = 0$$

where $u = xyz$

$v = x+y+z$

PDE is of the form $Pp + Qq = R$.

where $P = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} = \begin{vmatrix} xz & xy \\ 1 & 1 \end{vmatrix} = xz - xy$

$= x(z-y)$

$$\Rightarrow \boxed{P = x(z-y)}$$

Now $Q = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} = \begin{vmatrix} xy & yz \\ 1 & 1 \end{vmatrix} = xy - yz$

$= y(x-z)$

$$\Rightarrow \boxed{Q = y(x-z)}$$

and $R = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} yz & xz \\ 1 & 1 \end{vmatrix} = yz - xz$

$= z(y-x)$.

$$\Rightarrow \boxed{R = z(y-x)}$$

$\therefore$ PDE is of the form, $Pp + Qq = R$

$$\Rightarrow x(z-y)p + y(x-z)q = z(y-x)$$

which is the required PDE.

(2) Find PDE by eliminating arbitrary functions

$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

<u>Soln</u>:

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

$$\Rightarrow z - y^2 = 2f\left(\frac{1}{x} + \log y\right)$$

$$\frac{z-y^2}{2} = f\left(\frac{1}{x} + \log y\right)$$

$$\Rightarrow U = f(V)$$

where $U = \frac{z-y^2}{2}$

$V = \frac{1}{x} + \log y$

PDE is of the form $Pp + Qq = R$

where $P = \begin{vmatrix} U_y & U_z \\ V_y & V_z \end{vmatrix} = \begin{vmatrix} -y & \frac{1}{2} \\ \frac{1}{y} & 0 \end{vmatrix} = 0 - \frac{1}{2y} = \frac{-1}{2y}$

$$\boxed{P = \frac{-1}{2y}}$$

$Q = \begin{vmatrix} U_z & U_x \\ V_z & V_x \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{x^2} \end{vmatrix} = \frac{-1}{2x^2}$

$$\boxed{Q = \frac{-1}{2x^2}}$$

$R = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = \begin{vmatrix} 0 & -y \\ \frac{-1}{x^2} & \frac{1}{y} \end{vmatrix} = 0 - \frac{y}{x^2} = \frac{-y}{x^2}$

$$\boxed{R = -\frac{y}{x^2}}$$

$\therefore$ PDE is $Pp + Qq = R$

$$\Rightarrow \frac{-1}{2y}p + \frac{-1}{2x^2}q = \frac{-y}{x^2}$$

$$\Rightarrow \frac{p}{2y} + \frac{q}{2x^2} = \frac{y}{x^2}$$

$$\Rightarrow \frac{p}{y} + \frac{q}{x^2} = \frac{2y}{x^2}$$

$$\Rightarrow \frac{px^2 + qy}{yx^2} = \frac{2y}{x^2}$$

$$\Rightarrow px^2 + qy = 2y^2$$ which is required PDE.

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3 Form the PDE by eliminating arbitrary functions $f(x+y+z,\ x^2+y^2+z^2) = 0$

Soln:

$f(u,v) = 0$

$f(x+y+z,\ x^2+y^2+z^2) = 0$

$u = x+y+z$

$v = x^2+y^2+z^2$

PDE is of the form $Pp + Qq = R$

where $P = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2y & 2z \end{vmatrix} = 2z - 2y = 2(z-y)$

$$\boxed{P = 2(z-y)}$$

$Q = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2z & 2x \end{vmatrix} = 2x - 2z = 2(x-z)$

$$\boxed{Q = 2(x-z)}$$

$R = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = 2y - 2x = 2(y-x)$

$$\Rightarrow \boxed{R = 2(y-x)}$$

$\therefore$ PDE is $Pp + Qq = R$

$$\Rightarrow 2(z-y)p + 2(x-z)q = 2(y-x)$$

$$\Rightarrow (z-y)p + (x-z)q = (y-x)$$

SOLUTION OF A PDE

Linear equations of first order.

consider a PDE which is linear in p, q, r is of the form $Pp + Qq = R$, where P, Q, R are the functions of x, y, z. This is called Lagrange's linear equation which is of order one.

Method for Solving Lagrange's Linear eqn.

① METHOD OF GROUPING

Form the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. This is called Lagrange's Auxillary equation or Subsidiary equation. and then a solution can be obtained by using usual methods.

$\phi(u,v) = 0$, u and v are two indt solutions

$c_1 = U(x,y,z) = U$

$c_2 = V(x,y,z) = V$

(1) Solve $\frac{y^2 z}{x} p + xzq = y^2$ by method of grouping.

Soln:

Rewriting the given equation as,

$$y^2 z p + x^2 z q = y^2 x$$

It is of the form $Pp + Qq = R$

$$P = y^2 z, \quad Q = x^2 z, \quad R = y^2 x$$

The Auxiliary equations are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie $$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

The first two fractions give,

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$$

$$\Rightarrow \quad x^2 \cancel{z}\, dx = y^2 \cancel{z}\, dy$$

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} + c_1 = \frac{y^3}{3} + c_2$$

$$\frac{x^3}{3} - \frac{y^3}{3} = c_1$$

$$x^3 - y^3 = c_1, \text{ where } c_1 = U(x, y, z)$$

$\therefore$ first solution is $\boxed{U = x^3 - y^3}$

Now 1^{st} and 3^{rd} relations give,

$$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$$y^2 x \, dx = y^2 z \, dz$$

$$\Rightarrow \int x \, dx = \int z \, dz$$

$$\Rightarrow \frac{x^2}{2} + c_1 = \frac{z^2}{2} + c_2$$

$$\frac{x^2 - z^2}{2} = c_2$$

$$\boxed{x^2 - z^2 = \frac{c}{2}}$$

$\therefore$ second solution is $\boxed{V = x^2 - z^2}$

general solution is $\phi(u, v) = 0$

$\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$ which is the required solution.

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(2) Solve by Lagrange's 1st method,

$$Pz - Qz = z^2 + (x+y)^2$$

Soln:

It is of the form $Pp + Qq = R$.

$P = z$

$Q = -z$

$R = z^2 + (x+y)^2$

Auxillary eqn is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

Taking 1st and 2nd relations,

$$\frac{dx}{z} = \frac{dy}{-z}$$

$\cancel{z}\, dx = \cancel{z}\, dy$

$\int -dx = \int dy$

$-x = y + c_1$

$\Rightarrow \boxed{c_1 = x + y.}$

$\therefore$ 1st solution is $\boxed{U = x+y.}$

Now take 1st and 3rd relations,

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

$$\Rightarrow \int z^2+(x+y)^2\, dx = \int z\, dz.$$

$$\int dx = \int \frac{z}{z^2+(x+y)^2}\, dz$$

$$\int dx = \int \frac{z}{z^2+c_1^2}\, dz \qquad \text{where } c_1 = x+y.$$

$$\text{ie } x = \int \frac{dt}{2t} \qquad \text{put, } t = z^2+c_1^2$$

$dt = 2z\, dz$

$\Rightarrow z\, dz = \frac{dt}{2}$

$$\Rightarrow x = \frac{1}{2}\log t + \frac{c}{2}$$

$$2x = \log t + c_2$$

$$2x = \log(z^2+c_1^2) + c_2.$$

$$\Rightarrow \boxed{2x - \log(z^2+c_1^2) = c_2}$$

2nd solution is $\boxed{V = 2x - \log(z^2+c_1^2)}$

general soln is $\phi(u,v)=0$ ie $\phi\left(x+y,\ 2x-\log(z^2+c_1^2)\right)=0$

(3) Solve $p - 2q = 3x^2 \sin(y+2x)$ by method of grouping.

Soln:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad -(1)$$

$P = 1$

$Q = -2$

$R = 3x^2 \sin(y+2x)$

$\therefore$ A.E (1) is $\quad \frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y+2x)}$

from 1st and 2nd, $\quad \frac{dx}{1} = \frac{dy}{-2}$

$$\Rightarrow \int -2\,dx = \int 1\,dy$$

$$-2x = y + c_1$$

$$\Rightarrow \boxed{c_1 = y+2x}$$

$$\therefore \boxed{U = y+2x}$$

from 1st and 3rd relation,

$$\frac{dx}{1} = \frac{dz}{3x^2 \sin(y+2x)}$$

$$3x^2 \sin(y+2x)\,dx = dz$$

$$\int 3x^2 \sin c_1 \, dx = \int dz \qquad \text{where } c_1 = y+2x$$

$$\sin c_1 \frac{3x^3}{3} = z + c_2$$

$$\Rightarrow \boxed{x^3 \sin c_1 - z = c_2}$$

$\therefore$ $\boxed{v = x^3 \sin c_1 - z}$

$\therefore$ general solution is $\phi(u, v) = 0$

$$\phi\left(2x+y,\ x^3 \sin c_1 - z\right) = 0$$

II METHOD :- Method of Multiplying.

Let the auxillary equation be $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Let l, m, n are the functions of x, y, z or constants

l, m, n are chosen in such a way that

$lP + mQ + nR = 0 \Rightarrow$ we get 1st solution.

similarly choose another set of multiplier l, m, n.

we get second solution by integrating

$$l\,dx + m\,dy + n\,dz = 0$$

$\therefore$ general solution is $\phi(u, v) = 0$.

(1) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$ by method of multiplying

Soln:

$$Pp + Qq = R.$$

$P = x^2(y-z)$

$Q = y^2(z-x)$

$R = z^2(x-y)$

Using the method of multiplier,

we choose l, m, n such that $lP + mQ + nR = 0$

Taking the multiplier as,

$$l = \frac{1}{x^2}, \quad m = \frac{1}{y^2}, \quad n = \frac{1}{z^2}$$

$$lP + mQ + nR = 0$$

$$\frac{1}{x^2} \times x^2(y-z) + \frac{1}{y^2} \times y^2(z-x) + \frac{1}{z^2} \times z^2(x-y)$$

$$= y - z + z - x + x - y$$

$$= \underline{\underline{0}}$$

$$\Rightarrow l\,dx + m\,dy + n\,dz = 0$$

$$\int \frac{1}{x^2}\,dx + \int \frac{1}{y^2}\,dy + \int \frac{1}{z^2}\,dz = 0$$

$$\int x^{-2}\,dx + \int y^{-2}\,dy + \int z^{-2}\,dz = 0$$

$$\frac{x^{-1}}{-1} + \frac{y^{-1}}{-1} + \frac{z^{-1}}{-1} + c_1 = 0$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$$

$$\therefore \boxed{U = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

To find second solution, choose another set of multipliers ~~as~~, l, m, n as

$$l = \frac{1}{x}, \quad m = \frac{1}{y}, \quad n = \frac{1}{z}$$

$$\Rightarrow lP + mQ + nR = 0$$

$$\frac{1}{x} \times x^2(y-z) + \frac{1}{y} \times y^2(z-x) + \frac{1}{z} \times z^2(x-y)$$

$= x(y-z) + y(z-x) + z(x-y)$

$= xy - xz + yz - yx + zx - zy$

$= 0$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow \int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy + \int \frac{1}{z}\,dz = 0$

$\log x + \log y + \log z = c_2$

$\Rightarrow \boxed{c_2 = xyz}$

$\therefore \quad V = xyz.$

general solution is $\phi(u, v) = 0$

$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \; xyz\right) = 0$

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(2) Solve $(y-z)p + (x-y)q = z-x$ by the method of multipliers.

Soln:

$P = y - z$

$Q = x - y$

$R = z - x$

choose multipliers l, m, n as,

$l = 1, \quad m = 1, \quad n = 1$

Such that $lP + mQ + nR = 0$

$= 1\cdot(y-z) + 1\cdot(x-y) + 1\cdot(z-x) =$

$= y - z + x - y + z - x$

$= \underline{\underline{0}}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\int 1\,dx + \int 1\,dy + \int 1\,dz = 0$

$$\boxed{x+y+z=c_1}$$

$\therefore\ u = x+y+z$

choose another set of multipliers l, m, n as,

$l = x \quad m = z \quad n = y$

such that $lP + mQ + nR = 0$

$$= x(y-z) + z(x-y) + y(z-x)$$

$$= \cancel{xy} - \cancel{xz} + \cancel{zx} - \cancel{zy} + \cancel{yz} - \cancel{yx}$$

$$= 0$$

$\Rightarrow\ l\,dx + m\,dy + n\,dz = 0$

$$\int x\,dx + \int z\,dy + \int y\,dz = 0$$

$$\frac{x^2}{2} + zy + yz = c_2$$

$$\frac{x^2}{2} + 2zy = c_2$$

$$\therefore\ \boxed{c_2 = x^2 + 4zy}$$

$$V = x^2 + 4zy$$

$\therefore$ general solution is $\phi(u, v) = 0$

$$\phi\left(x+y+z,\ x^2+4zy\right) = 0.$$

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3) Solve $(mz-ny)p + (nx-lz)q = ly-mx$ by method of multiplying.

<u>soln</u>:

$$P = mz - ny$$

$$Q = nx - lz$$

$$R = ly - mx$$

choose multipliers l, m, n as,

$l = x, \quad m = y, \quad n = z$

Such that $lP + mQ + nR = 0$

$$\Rightarrow x(mz - ny) + y(nx - lz) + z(ly - mx)$$

$$= \cancel{xmz} - \cancel{xny} + \cancel{ynx} - \cancel{ylz} + \cancel{zly} - \cancel{zmx}$$

$$= 0$$

$$\Rightarrow l\,dx + m\,dy + n\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$\Rightarrow \boxed{c_1 = x^2 + y^2 + z^2}$$

$$\therefore U = x^2 + y^2 + z^2$$

To find second solution, choose another set of multipliers l, m, n as,

$l = l, \quad m = m, \quad n = n$

$$\Rightarrow lP + mQ + nR = 0$$

$$= l(mz - ny) + m(nx - lz) + n(ly - mx)$$

$$= \cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{mlz} + \cancel{nly} - \cancel{nmx}$$

$$= \underline{\underline{0}}$$

$$\Rightarrow l\,dx + m\,dy + n\,dz = 0$$

$$\int l\,dx + \int m\,dy + \int n\,dz = 0$$

$$\Rightarrow \boxed{lx + my + nz = c_2}$$

$$\therefore V = lx + my + nz$$

general solution is $\phi(u,v) = 0$

$$\Rightarrow \phi\left(x^2+y^2+z^2,\ lx+my+nz\right) = 0$$

METHOD OF SEPERATION OF VARIABLES

The method of seperation of variable is applicable for a large number of linear homogenous equation where all the terms of the PDE contains dependent values. This method reduces a partial differential equation in inde to a differential eqn.

Suppose that the given PDE contains 'n' independent variables $x_1, x_2, \ldots x_n$ and one dependent variable 'U'. Then solution is of the form

$$U(x_1, x_2 \ldots x_n) = X_1(x_1) \cdot X_2(x_2) \cdot \ldots\ldots X_n(x_n) \quad \text{— (1)}$$

where X_i is a fun. of x_i only.

Substitute (1) in given PDE, we get 'n' ordinary unknown function X_i then solve it by usual method.

Q) Solve by the method of variation of parameters.

i) $x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0$

Soln:

Assume general solution is $U = X(x) \cdot Y(y)$.

$$U = XY \quad — ①$$

X is a function of x and Y is a fun. of y.

we've $x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0 \quad — ②$

from ①, $\frac{\partial u}{\partial x} = X'Y = \frac{dX}{dx} Y$

$\frac{\partial u}{\partial y} = XY' = X \cdot \frac{dY}{dy}$

Substituting these in eqn ②, we get

② $\Rightarrow x X'Y - 2y XY' = 0$

$\Rightarrow x X'Y = 2y XY'$

$\Rightarrow \frac{xX'}{X} = \frac{2yY'}{Y}$

This can be true ~~only~~ when each side is equal to a constant which is called seperation constant

$$\Rightarrow \frac{xX'}{X} = \frac{2yY'}{Y} = k.$$

ie $\frac{xX'}{X} = k$

$\Rightarrow xX' = kX$

$x \frac{dX}{dx} = kX$

Also $\frac{2yY'}{Y} = k$

$\Rightarrow 2yY' = Yk$

$\Rightarrow 2y \frac{dY}{dy} = Y \cdot k$

$$\Rightarrow \frac{dx}{x} = k\frac{dx}{x}$$

$$\Rightarrow \log x = k\log x + c_1$$

$$\Rightarrow \log x = \log x^k + \log c_1$$

$$\Rightarrow \log x = \log(x^k c_1)$$

$$\Rightarrow \boxed{x = x^k c_1} \quad — ③$$

$$2\frac{dy}{y} = k\frac{dy}{y}$$

ie $2\log y = k\log y + c_2$

ie $\log y = \frac{k}{2}\log y + c_2$

$$\log y = \log y^{k/2} + \log c_2$$

$$\log y = \log(y^{k/2} \cdot c_2)$$

$$\Rightarrow \boxed{y = y^{k/2} c_2} \quad — ④$$

$\therefore$ solution $U = xy$

$$\Rightarrow U = (x^k c_1) \times (y^{k/2} c_2)$$

$$= c x^k y^{k/2} \quad [\because c = c_1 \times c_2]$$

ie $U = c x^k y^{k/2}$

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② Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ using method of seperation of variables.

Soln:-

$$U = xy \quad — ①$$

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad — ②$$

from ①, we've $\frac{\partial u}{\partial x} = x'y$

$$\frac{\partial^2 u}{\partial x^2} = x''y$$

$$\frac{\partial u}{\partial y} = xy'$$

Substituting these in ②, we get

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$X''Y - 2X'Y + XY' = 0$$

$$Y[X'' - 2X'] + XY' = 0$$

$$Y[X'' - 2X'] = -XY'$$

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y}$$

Using seperation constant, k.

$$\frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k$$

Now, $\frac{X'' - 2X'}{X} = k$

$$\Rightarrow X'' - 2X' - Xk = 0$$

This is ODE.

ie $[D^2 - 2D' - k]X = 0$

A.E is $m^2 - 2m - k = 0$.

~~$X = c_1 e^{tt}$~~ $\Rightarrow m = 1 \pm \sqrt{1+k}$

$$\Rightarrow X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

Now $-\frac{Y'}{Y} = k$

$$-Y' = kY$$

$$-\frac{dy}{dy} = kY$$

$$-\frac{dy}{Y} = k\,dy$$

Also, ~~$-\frac{Y'}{Y}$~~ $-\frac{Y'}{Y} = k$

$\Rightarrow$ solution is $c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+2 \pm \sqrt{4 + 4k}}{2}$$

$m = 1 \pm \sqrt{1+k}$, m_1, m_2

$$\boxed{X = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$$-ky = \log e^{-ky}$$

$$-\log y = ky + c_2 \qquad \angle \therefore \; -ky = \log e^{-ky}$$

$$\log y = -ky + c_2$$

$$\log y = \log e^{-ky} + \log c_2 .$$

$$\log y = \log\left(e^{-ky} \cdot c_2\right)$$

$$\boxed{y = e^{-ky} c_2}$$

$\therefore$ soln is, $u = xy$.

$$\Rightarrow u = \left[c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right] \times e^{-ky} c_2$$

$$= \left(c_1 e^{(1+\sqrt{1+k})x} \times c_2 e^{-ky}\right) + \left(c_2 e^{(1-\sqrt{1+k})x} \times e^{-ky} c_3\right)$$

$$= e^{-ky}\left[a e^{(1+\sqrt{1+k})x} + b e^{(1-\sqrt{1+k})x}\right]$$

$$= e^{-ky}\left[a e^{(1+\sqrt{1+k})x} + b e^{(1-\sqrt{1+k})x}\right]$$

where $a = c_1 c_2$

$b = c_2 c_3$

Q) Solve by the method of seperation of variables. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

where $u(x,0) = 6e^{-3x}$

Soln:-

Assume the solution $U = XT$ — ①

ie $U(x,t) = X(x).T(t)$.

from ①, we've $\frac{\partial u}{\partial x} = X'T$

$\frac{\partial u}{\partial t} = XT'$

substituting this in $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ — ②

we get $X'T = 2XT' + XT$

$\Rightarrow X'T - 2XT' - XT = 0$

$(X'-X)T - 2XT' = 0$

$\Rightarrow (X'-X)T = 2XT'$

$\Rightarrow \frac{(X'-X)}{X} = \frac{2T'}{T}$

Using seperation constant k,

$\frac{(X'-X)}{X} = \frac{2T'}{T} = k.$

$\therefore \frac{X'-X}{X} = k$

$\Rightarrow X'-X = Xk$

$X' - X - Xk = 0$

$\Rightarrow X' - X(1+k) = 0$

$X' = X(1+k)$

$\frac{X'}{X} = (1+k)$

Now, $\frac{2T'}{T} = k$

$\frac{2}{T}\frac{dT}{dt} = k$

$\Rightarrow \frac{dT}{T} = \frac{1}{2}k\,dt$

$\Rightarrow \log T = \frac{1}{2}kt + c_2$

$\Rightarrow \log T = \log e^{\frac{1}{2}kt} + \log c_2$

$$\Rightarrow \frac{dX}{X\,dx} = 1+k$$

$$\Rightarrow \frac{dX}{X} = (1+k)\,dx$$

$$\Rightarrow \log X = (1+k)x + c_1$$

$$\Rightarrow \log X = \log e^{(1+k)x} + \log c_1$$

$$\Rightarrow \log X = \log (e^{(1+k)x} \cdot c_1)$$

$$\Rightarrow \boxed{X = e^{(1+k)x} \cdot c_1}$$

$$\Rightarrow \log T = \log e^{\frac{1}{2}kt} + \log c_2$$

$$\log T = \log [e^{\frac{1}{2}kt} \cdot c_2]$$

$$\Rightarrow T = e^{\frac{1}{2}kt} \cdot c_2$$

$$\Rightarrow \boxed{T = e^{\frac{1}{2}kt} \cdot c_2}$$

Thus solution is,

$$U = XT$$

ie $U(x,t) = e^{(1+k)x} \cdot c_1 \cdot e^{\frac{1}{2}kt} \cdot c_2$ — (3)

given, $U(x,0) = 6e^{-3x}$ — (4)

from (3) & (4) $\Rightarrow U(x,0) = e^{(1+k)x} \cdot c_1 \cdot c_2 \cdot e^{0}$.

$$\Rightarrow U(x,0) = ce^{(1+k)x} = 6e^{-3x}$$

Comparing $\Rightarrow$ $c = 6$ and $(1+k) = -3$

$\Rightarrow$ $c = 6$ and $k = -4$.

$\therefore$ (3) $\Rightarrow$ $U = e^{(1+-4)x} \times 6 \times e^{\frac{1}{2} \cdot -4t}$

$$= 6e^{-3x} \cdot e^{-2t}$$

$= 6e^{-(3x+2t)}$ which is the required solution.

Non linear equations of first order.

CHARPIT'S METHOD :-

consider $f(x, y, z, p, q) = 0$.

Since z depends on x & y,

$$\boxed{dz = p\,dx + q\,dy}$$

Auxillary equation is,

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

Q) Solve $2xz - px^2 - 2qxy + pq = 0$

Solution:

$f(x, y, z, p, q) = 0$

$\Rightarrow 2xz - px^2 - 2qxy + pq = 0$ — (1)

A.E is,

$$\frac{dx}{-(-x^2+q)} = \frac{dy}{-(-2xy+p)} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)} =$$

$$= \frac{dp}{(2z - 2\cancel{px} - 2qy) + p(\cancel{2x})} = \frac{dq}{-2qx + q(2x)}$$

from last 2 equations $\Rightarrow \dfrac{dp}{2z - 2qy} = \dfrac{dq}{0}$

$$\Rightarrow dq = 0$$

$$\Rightarrow \boxed{q = a}$$

substitute $q = a$ in eqn (1),

$$2xz - px^2 - 2axy + pa = 0$$

$\Rightarrow -px^2 + pa = 2axy - 2xz$

$\Rightarrow p[a - x^2] = 2x(ay - z)$

$\Rightarrow p = \dfrac{2x(ay-z)}{a-x^2}$

ie $\boxed{p = 2x\dfrac{(z-ay)}{x^2-a}}$

$\therefore dz = p\,dx + q\,dy.$

$\Rightarrow dz = \dfrac{2x(z-ay)}{x^2-a}dx + a\,dy.$

ie $dz - a\,dy = \dfrac{2x(z-ay)}{x^2-a}dx$

$\Rightarrow \dfrac{dz - a\,dy}{z-ay} = \dfrac{2x}{x^2-a}dx$

$= \dfrac{d(z-ay)}{z-ay} = \dfrac{2x}{x^2-a}dx$

$\boxed{\dfrac{d}{dx}\log(x^2-a) = \dfrac{1}{x^2-a}\times 2x}$

$\Rightarrow \log(z-ay) = \log(x^2-a) + b$

$\Rightarrow \log(z-ay) = \log(x^2-a) + \log b.$

ie $\log(z-ay) = \log(b.(x^2-a))$

$\Rightarrow z - ay = b(x^2-a)$

ie $z = b(x^2-a) + ay.$

② Solve $(p^2+q^2)y = qz$ by charpit's method

Soln:

$$f(x,y,z,p,q)=0 \Rightarrow (p^2+q^2)y - qz = 0 \quad \text{— ①}$$

Charpit's Auxiliary eqn is,

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{-2py} = \frac{dy}{-2qy} = \frac{dz}{-p(2py)-q(2qy)} = \frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2}$$

$\Rightarrow$ from last two eqns,

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

Integrating $\int p\,dp = \int -q\,dq$

$$\frac{p^2}{2} = \frac{-q^2}{2} + c$$

$$p^2+q^2 = c^2 \quad \text{— ②}$$

$$① \Rightarrow (p^2+q^2)y - qz = 0$$

$$c^2y - qz = 0$$

$$qz = c^2y \Rightarrow q = \frac{c^2y}{z}$$

$$\therefore p^2 = c^2 - q^2 \quad (\text{from ②})$$

$$p^2 = c^2 - \frac{c^4y^2}{z^2}$$

$$= \frac{c^2z^2 - c^4y^2}{z^2}$$

$$\Rightarrow p^2 = \sqrt{2}$$

$$\Rightarrow p^2 = c^2\left(\frac{z^2 - y^2c^2}{z^2}\right)$$

$$p = \frac{c}{z}\sqrt{z^2 - y^2c^2}$$

$$\therefore \; dz = p\,dx + q\,dy$$

$$dz = \frac{c\sqrt{z^2 - c^2y^2}}{z}\,dx + \frac{c^2y}{z}\,dy$$

$$z\,dz = c\sqrt{z^2 - c^2y^2}\,dx + c^2y\,dy$$

$$\frac{z\,dz - c^2y\,dy}{\sqrt{z^2 - c^2y^2}} = c\,dx$$

$$\frac{d(z - c^2y)}{\sqrt{z - c^2y^2}} = c\,dx$$

MODULE - II

WAVE EQUATION

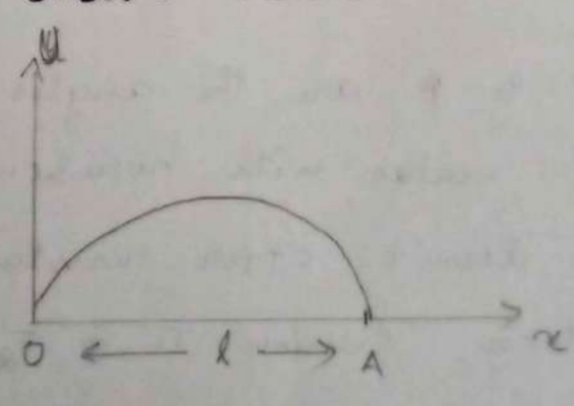
one of the most fundamental and common phenomina that occurs in nature is the phenomenon of wave motion.

The vibration of stretched string (one dimensional wave eqn] is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

DERIVATION OF ONE DIMENSIONAL WAVE EQUATION

proof:

Consider a uniform elastic string of length 'l' stretched tightly between two end points 'O' and 'A' as shown below.



Assumption

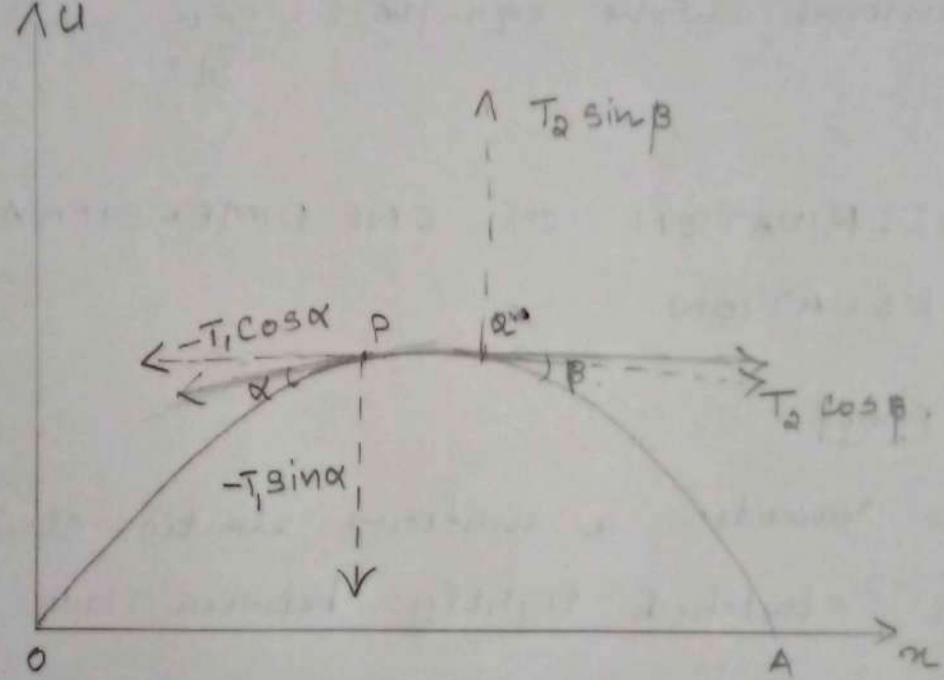
1) The motion take place entirely in xu plane

2) Each particle moving in vertical direction

3) The string is perfectly flexible and does not offer resistance to bending.

4) The tension 'T' in the string is so large that the force due to weight of the string may be neglected.

5) The displacement 'u' and slope $\frac{\partial u}{\partial x}$ are small.



Let α & β are the angles that the tension T_1 & T_2 makes with horizontal path. Since the string doesn't offer resistance to bending, by assumption, tension T_1 & T_2 at P & Q are tangential to the curve. Since there is no motion in the horizontal direction, the total horizontal force acting on PQ is zero.

ie, $-T_1 \cos\alpha + T_2 \cos\beta = 0$

$$\Rightarrow T_1 \cos\alpha = T_2 \cos\beta = T \text{ (say)} \quad \text{— (1)}$$

Let the mass per unit length of the string = m

$\therefore$ mass of PQ = $m \cdot \Delta x$ [mass $\times$ length]

by Newton's second law of motion, the total force acting in vertical direction,

$$\Rightarrow -T_1 \sin\alpha + T_2 \sin\beta = (m \cdot \Delta x) \cdot \frac{\partial^2 u}{\partial t^2}$$

$\therefore F = m \cdot a$

a = acceleration

$= \frac{\partial^2 u}{\partial t^2}$

dividing through out by T.

$$\Rightarrow \frac{-T_1 \sin\alpha}{T} + \frac{T_2 \sin\beta}{T} = \frac{(m \cdot \Delta x) \cdot \frac{\partial^2 u}{\partial t^2}}{T} \quad \text{— (2)}$$

from eqn (1), we get $T_1 \cos\alpha = T$ and $T_2 \cos\beta = T$

Substituting in eqn (2),

$$= \frac{-T_1 \sin\alpha}{T_1 \cos\alpha} + \frac{T_2 \sin\beta}{T_2 \cos\beta} = \frac{m \cdot \Delta x}{T} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= -\tan\alpha + \tan\beta = \frac{m \cdot \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$= \tan\beta - \tan\alpha = \left(\frac{m \cdot \Delta x}{T}\right) \cdot \frac{\partial^2 u}{\partial t^2}$$

Since $\tan\alpha$ = slope of P = $\left.\frac{\partial u}{\partial x}\right)_x$

and $\tan\beta$ = slope of Q = $\left.\frac{\partial u}{\partial x}\right]_x$

$$\Rightarrow \text{slop of Q} - \text{slope of P} = \left(\frac{m \cdot \Delta x}{T.}\right) \cdot \frac{\partial^2 u}{\partial t^2}$$

dividing by Δx both sides.

$$\Rightarrow \left[\frac{\partial u}{\partial x}\right]_{x+\Delta x} - \left[\frac{\partial u}{\partial x}\right]_{x} = \frac{(m \cdot \Delta x)}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left[\frac{\partial u}{\partial x}\right]_{x}}{\Delta x} = \left(\frac{m \cdot \Delta x}{T \cdot \Delta x}\right) \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{m}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

taking $\Delta x \to 0$ on both sides

$$\lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{m}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{m}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$\because \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$

and $\frac{T}{m} = c^2$

where $c^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length}}$

One dimensional wave eqn is,

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Solution of wave equation by the method of seperation of variables.

Soln:

Wave eqn is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ — ①

By the method of seperation of variables, general solution is $U = XT$ — ②

Now from ①, $\frac{\partial u}{\partial t} = XT'$

$$\frac{\partial^2 u}{\partial t^2} = XT''$$

$$\frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

substituting all these values in ①, we get

$$X T'' = c^2 X'' T$$

$$\frac{X''}{X} = \frac{T''}{T} \cdot \frac{1}{c^2}$$

Using the seperation constant, k,

$$\frac{X''}{X} = \frac{T''}{T} \cdot \frac{1}{c^2} = k$$

ie $\frac{X''}{X} = k$ and $\frac{T''}{T} \frac{1}{c^2} = k$

Now,

$$\frac{X''}{X} = k$$

$$X'' - Xk = 0$$

also, $\frac{T''}{T} \cdot \frac{1}{c^2} = k$

$$T'' - Tc^2 k = 0$$

$X'' - Xk = 0$

$(D^2 - k)X = 0$

A.E $\Rightarrow \lambda^2 - k = 0$

$\boxed{\lambda^2 = k}$ — (a)

$T'' - kc^2T = 0$

$(D^2 - kc^2)T = 0$

A.E $\Rightarrow \lambda^2 - kc^2 = 0$

$\boxed{\lambda^2 = kc^2}$ — (b)

These two equations depends on values of k.

$\therefore$ There are three cases.

Case I: when $k = 0$.

(a) $\Rightarrow \lambda^2 = k$

when $k=0 \Rightarrow \lambda^2 = 0$

$\Rightarrow \lambda = 0, 0$

$\therefore \boxed{X = [c_1 + c_2 x]}$

(b) $\Rightarrow \lambda^2 = kc^2$

when $k=0 \Rightarrow \lambda^2 = 0$.

$\Rightarrow \lambda = 0, 0$

$\therefore \boxed{T = [c_3 + c_4 t]}$

$\therefore$ general solution $U = XT$

$\Rightarrow \boxed{U = [c_1 + c_2 x][c_3 + c_4 t]}$ — (A)

Case II: when k is positive, ie $k = p^2$

(a) $\Rightarrow \lambda^2 = k$

when k is +ve, $k = p^2$

$\Rightarrow \lambda^2 = p^2$

$\Rightarrow \lambda = \pm p$

$\therefore \boxed{X = c_1 e^{px} + c_2 e^{-px}}$

(b) $\Rightarrow \lambda^2 = kc^2$

when $k = p^2$,

$\lambda^2 = p^2c^2$

$\Rightarrow \lambda = \pm pc$

$\therefore \boxed{T = c_3 e^{pct} + c_4 e^{-pct}}$

$\therefore$ general soln is $U = XT$

$$\Rightarrow \boxed{U = \left[c_1 e^{px} + c_2 e^{-px}\right]\left[c_3 e^{pct} + c_4 e^{-pct}\right]} \quad \text{— (B)}$$

Case III :- when k is negative ie $k = -p^2$

ⓐ $\Rightarrow \lambda^2 = k$

when k is -ve, $k = -p^2$

$\lambda^2 = -p^2$

$\Rightarrow \lambda = \pm ip$

$\therefore \boxed{X = c_1 \cos px + c_2 \sin px}$

ⓑ $\Rightarrow \lambda^2 = kc^2$

when k is -ve, $k = -p^2$

$\lambda^2 = -p^2c^2$

$\lambda = \pm ipc$

$\boxed{T = c_3 \cos pct + c_4 \sin pct}$

$\therefore$ general soln is $U = XT$

$$\Rightarrow \boxed{U = \left[c_1 \cos px + c_2 \sin px\right]\left[c_3 \cos pct + c_4 \sin pct\right]} \quad \text{— (C)}$$

Since we are dealing with a problem on vibrations, y must be a periodic function of x and t.

$\therefore$ Solution must involve trignometric terms.

$\therefore$ soln of wave eqn is

$$\boxed{U = U(x,t) = \left[c_1 \cos px + c_2 \sin px\right]\left[c_3 \cos pct + c_4 \sin pct\right]}$$

Boundary conditions and Initial conditions for one diamensional wave equation.

Boundary conditions are

$U(0,t) = 0$ — ①

$U(l,t) = 0$ — ②

Initial conditions are

$\frac{\partial u}{\partial t}$ at $(x,0) = 0$ — ③

$U(x,0) = f(x)$ — ④

The solution of wave equation is called vibration of a stretched string

ie solution is,

$U(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pct + c_4 \sin pct]$ — ⓐ

appling boundary and initial conditions, we get

Boundary condition :- (B C)

① $\Rightarrow U(0,t) = 0 \Rightarrow c_1 [c_3 \cos pct + c_4 \sin pct] = 0$

$\Rightarrow \boxed{c_1 = 0}$

② $\Rightarrow U(l,t) = 0 \Rightarrow U(x,t) = 0$ when $x = l$

$\Rightarrow U(l,t) = [c_1 \cos pl + c_2 \sin pl)][c_3 \cos pct + c_4 \sin pct] = 0$

$\Rightarrow$ Since $c_1 = 0$,

$U(l,t) = c_2 \sin pl [c_3 \cos pct + c_4 \sin pct] = 0$

$\Rightarrow c_2 \sin pl = 0$

$\Rightarrow \sin pl = 0$ since $c_2 \neq 0$, only $\sin pl = 0$

$\Rightarrow pl = n\pi$, since $\sin n\pi = 0$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

substituting the values $c_1 = 0$ and $p = \frac{n\pi}{l}$ in eq (a)

It becomes,

$$U(x,t) = \left[c_2 \sin \frac{n\pi}{l} x\right]\left[c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t\right]$$

combining the constants c_2, c_3 & c_4 as,

$$c_2 \cdot c_3 = a_n \quad \& \quad c_2 c_4 = b_n$$

$$\therefore U(x,t) = \sin \frac{n\pi}{l} x \left\{\left[c_2 c_3 \cos \frac{cn\pi}{l} t\right] + \left[c_2 c_4 \sin\left(\frac{cn\pi}{l}\right) t\right]\right\}$$

$$= \sin \frac{n\pi}{l} x \left[a_n \cos \frac{cn\pi}{l} t + b_n \sin \frac{cn\pi}{l} t\right]$$

adding up the solutions for different values of n,

we get,

$$\boxed{y = U(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{cn\pi}{l} t + b_n \sin \frac{cn\pi}{l} t\right] \sin \frac{n\pi}{l} x} \longrightarrow (b)$$

is also a solution.

Now applying initial conditions, IC

③ $\Rightarrow \frac{\partial u}{\partial t}(x,0) = 0$

we've

$$U(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos \frac{cn\pi}{l} t + b_n \sin \frac{cn\pi}{l} t\right] \sin \frac{n\pi}{l} x \quad \text{— (6)}$$

Now $\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = \sum_{n=1}^{\infty}\left[a_n -\sin \frac{cn\pi}{l} t \times \frac{cn\pi}{l} + b_n \cos \frac{cn\pi}{l} t \times \frac{cn\pi}{l}\right] \times \sin \frac{n\pi}{l} x = 0$

$$\Rightarrow \sum_{n=1}^{\infty}\left[b_n \frac{cn\pi}{l}\right] \cdot \sin \frac{n\pi}{l} x = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

applying boundary and Initial conditions,

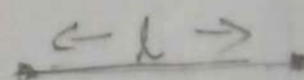
$$U(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos \frac{cn\pi}{l} t\right] \sin \frac{n\pi}{l} x.$$

④ $\Rightarrow U(x,0) = f(x)$

$\Rightarrow \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x = f(x)$, which represents half range sine series.

where $a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$

Q) A tightly stretched string of length 'l' fixed at both ends. Find the displacement $U(x,t)$ of the string, given an initial displacement $f(x)$ and initial velocity $g(x)$.



Soln:

one diamensional wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

general soln is,

$$U(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pct + c_4 \sin pct]$$

Boundary conditions are (BC)

① $U(0,t)=0 \Rightarrow \boxed{C_1=0}$

② $U(l,t)=0 \Rightarrow \boxed{p=\frac{n\pi}{l}}$

$\therefore$ Soln $U(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\frac{cn\pi}{l}t+b_n\sin\frac{cn\pi}{l}t\right]\sin\frac{n\pi}{l}x$

Initial conditions are (IC)

③ $\frac{\partial u}{\partial t}$ at $(x,0)=g(x)$

$\Rightarrow \frac{cn\pi}{l}\left[\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{l}\right)x\right]=g(x)$

which is also half range sine series.

$$* \quad \boxed{\Rightarrow b_n=\frac{2}{cn\pi}\int_0^l g(x)\cdot\sin\frac{n\pi}{l}x\,dx}$$

④ $\Rightarrow U(x,0)=f(x)$

$=\sum_{n=1}^{\infty}\left[a_n\sin\frac{n\pi}{l}x\right]$

where $\boxed{a_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx.}$

$\therefore$ ~~Initial~~ displacement,

$u(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\frac{cn\pi}{l}t+b_n\sin\frac{cn\pi}{l}t\right]\sin\frac{n\pi}{l}x$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\, \sin\frac{n\pi}{l}x \, dx$$

and $b_n = \dfrac{2}{cn\pi}\displaystyle\int_0^l g(x)\, \sin\frac{n\pi}{l}x \, dx$

NOTE:

※ (1) when the string is in equilibrium position, initial displacement $U(x,0) = f(x) = 0$

$$\Rightarrow f(x) = 0 \Rightarrow a_n = 0$$

(2) when Initial velocity is zero, then

$$\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0 \Rightarrow g(x) = 0 \Rightarrow b_n = 0$$

Problems :-

(Q1) A tightly stretched string of length 'l' with its fixed ends at $x=0$ & $x=l$ which execute transverse ~~direction~~ Motion starts with zero initial velocity by displacing the string in to the form $f(x) = k(x^2 - x^3)$. Find the displacement $U(x,t)$ at any time 't'.

Soln: One dimensional wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$U(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pct + c_4 \sin pct]$$

BC

(1) $U(x,0) = 0$

$$\Rightarrow \boxed{c_1 = 0}$$

② $u(l,t) = 0$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

IC :-

③ $\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0$ Since motion starts with

$\Rightarrow b_n = 0$ $\therefore$ zero velocity.

④ $u(x,0) = f(x)$

$$\Rightarrow \sum_{n=1}^{\infty}\left[a_n \sin \frac{n\pi}{l}x\right] = f(x)$$

$$\Rightarrow \sum a_n \sin \frac{n\pi x}{l} = k(x^2 - x^3)$$

where $a_n = \frac{2}{l}\int_0^l f(x) \cdot \sin \frac{n\pi}{l}x \, dx$

$$= \frac{2}{l}\int_0^l k(x^2 - x^3) \sin \frac{n\pi}{l}x \, dx$$

$$= \frac{2k}{l}\left\{\left[(x^2 - x^3) \cdot \frac{-\cos \frac{n\pi}{l}x}{\frac{n\pi}{l}} \times \cancel{\frac{n\pi}{l}}\right]_0^l - \int_0^l (2x - 3x^2) \frac{-\cos \frac{n\pi}{l}x}{\frac{n\pi}{l}} dx\right.$$

$$= \frac{2k}{l}\left\{-\frac{\left[l^2 - l^3\right]\cos n\pi}{\frac{n\pi}{l}} + \frac{1}{\frac{n\pi}{l}}\int_0^l (2x - 3x^2) \cos \frac{n\pi}{l}x \, dx\right.$$

$$= \frac{2k}{l}\left\{\left(-\frac{(-1)^n(l^2 - l^3)}{\frac{n\pi}{l}}\right) + \frac{1}{\frac{n\pi}{l}}\left\{\left[(2x - 3x^2) \cdot \frac{\sin \frac{n\pi}{l}x}{\frac{n\pi}{l}}\right]_0^l - \int_0^l (2 - 6x) \frac{\sin \frac{n\pi}{l}x}{\frac{n\pi}{l}}\right.\right.$$

$$= \frac{2k}{l}\left\{\left[-(-1)^n (l^2 - l^3) \frac{\times l}{n\pi}\right] + \frac{l^2}{n\pi}\left[\underbrace{(2l - 3l^2)\times 0 - 0}_{=0}\right] - \frac{l}{n\pi}\int_0^l (2 - 6x) \sin \frac{n\pi}{l}x \, dx\right.$$

$$= \frac{2k}{l}\left\{\left[-(-1)^n(l^2-l^3)\frac{l}{n\pi} + \frac{-l}{n\pi}\left[(2-6x)\cdot\frac{-\cos\frac{n\pi}{l}}{\frac{n\pi}{l}}\right]_0^l - \int_0^l -6 \cdot \frac{-\cos\frac{n\pi}{l}}{\frac{n\pi}{l}}\,dx\right.\right.$$

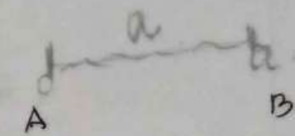
$$= \frac{2k}{l}\left\{\left[-(-1)^n(l^2-l^3)\frac{l}{n\pi} + \frac{l^2}{(n\pi)^2}\left[(2-6l)\cos n\pi - 2\cdot\cos 0\right] - \frac{6l}{n\pi}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l\right.\right.$$

$$= \frac{2k}{l}\left\{-(-1)^n(l^2-l^3)\frac{l}{n\pi} + \frac{l^2}{(n\pi)^2}\left[(2-6l)(-1)^n - 2\right)\right] - \frac{6l^2}{(n\pi)^2}\left[0\right]$$

$$a_n = \frac{2k}{l}\left\{\left[-(-1)^n(l^2-l^3)\frac{l}{n\pi} + \frac{l^2}{(n\pi)^2}(2-6l)(-1)^n - 2\right\}.\right.$$

$$\therefore U(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\frac{cn\pi}{l}t\right]\sin\frac{n\pi}{l}x$$

Q) A tightly vibrating string of length 'a' is stretched between two points. A & B. Initial displacement of each part of the string is zero. and the initial velocity at a distance x from 'a' is $kx(a-x)$. Find the form of string at any subsequent time. ?



Soln:

One diamensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

gen. soln is $U(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pct + c_4 \sin pct]$

BC

① $\cancel{\frac{\partial u}{\partial x}}$ $U(0,t) = 0 \Rightarrow c_1 = 0$

② $U(l,t) = 0 \Rightarrow p = \frac{n\pi}{a}$ where $l = a$

IC

③ $\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = g(x) = kx(a-x)$

④ $U(x,0) = f(x) = 0 \Rightarrow a_n = 0$

$\therefore$ Soln is, $U(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\frac{cn\pi}{a}t + b_n \sin\frac{cn\pi}{a}t\right] \sin\frac{n\pi}{a}x$

from ④, $U(x,t) = \sum_{n=1}^{\infty} \left[b_n \sin\frac{cn\pi}{a}t\right] \sin\frac{n\pi}{a}x$ —— ⓐ

where $b_n = \dfrac{2}{cn\pi} \int_0^l g(x) \cdot \sin \dfrac{n\pi}{l} x \, dx$

since $l = a$, $b_n = \dfrac{2}{cn\pi} \int_0^a g(x) \cdot \sin \dfrac{n\pi}{a} x \, dx$

$$\Rightarrow b_n = \frac{2}{cn\pi} \int_0^a kx(a-x) \sin \frac{n\pi}{a} x \, dx$$

$$= \frac{2k}{cn\pi} \left\{ \left[(ax - x^2) \cdot \frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right]_0^a - \int_0^a (a-2x) \cdot \frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}} \, dx \right.$$

$$= \frac{2k}{cn\pi} \left\{ 0 + \frac{a}{n\pi} \int_0^a (a-2x) \cos \frac{n\pi}{a} x \, dx \right.$$

$$= \frac{2k}{cn\pi} \times \frac{a}{n\pi} \left\{ \left[(a-2x) \frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right]_0^a - \int_0^a -2 \cdot \frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}} \, dx \right.$$

$$= \frac{2k}{cn\pi} \times \frac{a}{n\pi} \left\{ 0 + \frac{2a}{n\pi} \int_0^a \sin \frac{n\pi}{a} x \, dx \right.$$

$$= \frac{2k}{cn\pi} \times \frac{a}{n\pi} \times \frac{2a}{n\pi} \left[\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}} \right]_0^a$$

$$= \frac{4a^2 k}{c(n\pi)^3} \left[-\left(\frac{\cos n\pi - 1}{\frac{n\pi}{a}} \right) \right]$$

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

$$= \frac{4a^3 k}{c(n\pi)^4} \left[-\left((-1)^n - 1 \right) \right] =$$

$$\therefore \boxed{b_n = \frac{4a^3 k}{c(n\pi)^4} \left[1 - (-1)^n \right]}$$

from (4) $\Rightarrow$ $a_n = 0$.

Initial disp. $= 0$ ie $U(x,0) = f(x) = 0$

$$\Rightarrow U(x,t) = \sum_{n=1}^{\infty} \left[b_n \sin \frac{cn\pi}{a} t \right] \sin \frac{n\pi}{a} x$$

$$\therefore \quad U(x,t) = \sum_{n=1}^{\infty} \left[\frac{4a^3k}{c(n\pi)^4} (1-(-1)^n) \right] \sin \frac{cn\pi}{a} t \; \sin \frac{n\pi}{a} x$$

(3) A tightly stretched string with fixed end points $x=0$ & $x=l$ is initialy at rest in equilibrium position. If it is set to be vibrating by giving to each of its points of a velocity $\lambda x(l-x) = g(x)$ (initial velocity) Find the displacement of the string at a distance x from one end at time t.

Soln:

one diamensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution is,

$$U(x,t) = [c_1 \cos px + c_2 \sin px][c_3 \cos pct + c_4 \sin pct]$$

BC

1) $U(0,t) = 0 \Rightarrow c_1 = 0$

2) $U(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$

$$\therefore U(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{cn\pi}{l} t + b_n \sin \frac{cn\pi}{l} t \right] \sin \frac{n\pi}{l} x$$

IC

(3) $\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = g(x) = \lambda x(l-x)$

$\Rightarrow b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi}{l}x\,dx$

(4) $u(x,0) = 0 \Rightarrow \phi(x) = 0$

$\Rightarrow a_n = 0$

$\therefore$ since given that string is in equilibrium position.

$\therefore u(x,t) = \sum_{n=1}^{\infty}\left[b_n \cos\frac{cn\pi}{l}t\right]\sin\frac{n\pi}{l}x$

Now (3) $\Rightarrow$ $b_n = \frac{2}{n\pi c}\int_0^l g(x)\cdot\sin\frac{n\pi}{l}x\,dx$

$\Rightarrow b_n = \frac{2}{n\pi c}\int_0^l \lambda x(l-x)\cdot\sin\frac{n\pi}{l}x\,dx$

$= \frac{2\lambda}{n\pi c}\int_0^l (lx-x^2)\sin\frac{n\pi}{l}x\,dx$

$= \frac{2\lambda}{n\pi c}\left\{\left[(lx-x^2)\frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right]_0^l - \int_0^l (l-2x)\frac{-\cos\frac{n\pi}{l}}{\frac{n\pi}{l}}\,dx\right.$

$= \frac{2\lambda}{n\pi c}\left\{0 + \int_0^l (l-2x)\frac{\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\,dx\right.$

$= \frac{2\lambda}{n\pi c}\times\frac{l}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi}{l}x\,dx$

$= \frac{2\lambda l}{n\pi c\cdot n\pi}\left\{\left[(l-2x)\cdot\frac{\sin\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right]_0^l - \int_0^l -2\,\frac{\sin\frac{n\pi}{l}x}{\frac{n\pi}{l}}\,dx\right.$

$\boxed{\sin n\pi = 0}$

$$\Rightarrow b_n = \frac{2\lambda l}{n\pi c \cdot n\pi}\left\{0 + \frac{2l}{n\pi}\int_0^l \sin\frac{n\pi}{l}x\,dx\right.$$

$$b_n = \frac{2\lambda l}{n\pi c \cdot n\pi \cdot n\pi}\,2l\left[\frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right]_0^l \qquad \because \cos n\pi = (-1)^n,\ \cos 0 = 1$$

$$= \frac{\cancel{4}\lambda l^2}{(n\pi)^3 c}\cdot\frac{l}{n\pi}\left[-(\cos n\pi - \cos 0)\right]$$

$$= \frac{\lambda l^3}{(n\pi)^4 \cdot c}\left[-((-1)^n - 1)\right]$$

$$b_n = \frac{\lambda l^3}{c(n\pi)^4}\left[1-(-1)^n\right]$$

$$\therefore \text{Soln},\ U(x,t) = \sum_{n=1}^{\infty}\left[b_n \sin\frac{cn\pi}{l}t\right]\cdot\sin\frac{n\pi}{l}x$$

$$\text{ie}\quad U(x,t) = \sum_{n=1}^{\infty}\frac{\lambda l^3}{c(n\pi)^4}\left[1-(-1)^n\right]\cdot\sin\frac{cn\pi}{l}t\cdot\sin\frac{n\pi}{l}x.$$

---

(4) A tightly stretched string of length 'a' with fixed ends is initially in equilibrium position. Find the displacement $U(x,t)$ of string if its ~~vibrating~~ by giving each of its points velocity $V_0 \sin\left(\frac{\pi x}{a}\right)$.

Soln:

one diamentional wave eqn is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

soln is $U(x,t) = \left[c_1 \cos px + c_2 \sin px\right]\left[c_3 \cos pct + c_4 \sin pct\right]$

BC

① $u(0,t) = 0 \Rightarrow c_1 = 0$

② $U(l,t) = 0 \Rightarrow p = \frac{n\pi}{a}$ $\quad \because l = a =$ length

IC

③ $\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = g(x) = V_0 \sin \frac{\pi}{a} x$

④ $U(x,0) = f(x) = 0$ $\quad \because$ since string is in equilibrium position.

$\Rightarrow a_n = 0.$

$\therefore U(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{cn\pi}{a} t + b_n \sin \frac{cn\pi}{a} t\right] \sin \frac{n\pi}{a} x$

from ④ $\Rightarrow a_n = 0$

$\Rightarrow U(x,t) = \sum_{n=1}^{\infty} \left[b_n \sin \frac{cn\pi}{a} t \quad \sin \frac{n\pi}{a} x\right]$

Therefore $b_n = \frac{2}{cn\pi} \int_0^l g(x) \cdot \sin \frac{n\pi}{a} x \, dx$

$\Rightarrow b_n = \frac{2}{cn\pi} \int_0^a V_0 \sin \frac{\pi}{a} x \quad \sin \frac{n\pi}{a} x \, dx$

for $n = 1$,

$b_n = \frac{2}{\pi c} \int_0^a V_0 \quad \sin \frac{\pi}{a} x \cdot \sin \frac{\pi}{a} x \, dx$

$= \frac{2V_0}{\pi c} \int_0^a \sin^2 \frac{\pi}{a} x \, dx$

$$\Rightarrow b_n = \frac{2V_0}{\pi c}\int_0^a \left[\frac{1-\cos\frac{2\pi}{a}x}{2}\right]dx \quad \angle \therefore \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\Rightarrow b_n = \frac{2V_0}{2\cdot\pi c}\int_0^a \left[1-\cos\frac{2\pi}{a}x\right]dx$$

$$\Rightarrow b_n = \frac{V_0}{\pi c}\left[x - \frac{\sin\frac{2\pi}{a}x}{\frac{2\pi}{a}}\right]_0^a$$

$$\Rightarrow b_n = \frac{V_0}{\pi c}\left[a - \frac{a}{2\pi}\left(\sin 2\pi - \sin 0\right)\right]$$

$$\Rightarrow b_n = \frac{V_0}{\pi c}\left[a - \frac{a}{2\pi}(0-0)\right] \quad \angle \therefore \sin 2\pi = 0, \ \sin 0 = 0$$

$$\Rightarrow \boxed{b_n = \frac{aV_0}{\pi c}}$$

$\therefore$ Soln is $U(x,t) = \sum_{n=1}^{\infty}\left[b_n \sin\frac{cn\pi}{l}t\right]\sin\frac{n\pi}{l}x$

ie, $U(x,t) = \sum_{n=1}^{\infty}\left[\frac{aV_0}{\pi c}\cdot\sin\frac{cn\pi}{l}t\right]\sin\frac{n\pi}{l}x$

---

(5) A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a\sin\frac{\pi x}{l}$ from which it is released at a time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by,

$$y(x,t) = a \sin\left(\frac{\pi x}{l}\right)\cdot \cos\left(\frac{\pi c t}{l}\right)$$

Soln:

The wave eqn on vibration of the string is given by,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution is $U(x,t) = y = \left[c_1 \cos px + c_2 \sin px\right]\left[c_3 \cos pct + c_4 \sin pct\right]$

BC

(1) $U(0,t) = y(0,t) = 0 \Rightarrow c_1 = 0$

(2) $U(l,t) = y(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$

IC

(3) $\frac{\partial u}{\partial t} = \frac{\partial y}{\partial t}\Big|_{(x,0)} = g(x) = 0$ $\quad$ ∵ since it is released at $t=0$.

$\Rightarrow b_n = 0$

(4) $U(x,0) = y(x,0) = f(x) = a \sin\frac{\pi x}{l} = y$

$$\Rightarrow a_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\frac{n\pi x}{l}\, dx$$

$$a_n = \frac{2}{l}\int_0^l y \sin\frac{n\pi x}{l}\, dx$$

$$\therefore U(x,t) = y(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right]\sin\frac{n\pi}{l}x$$

ie $\quad y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi c}{l}t \sin\frac{n\pi}{l}x$

where $a_n = \frac{2}{l}\int_0^l y \cdot \sin\frac{n\pi}{l}x\, dx$

$$a_n = \frac{2}{l}\int_0^l a \sin\frac{\pi x}{l} \cdot \sin\frac{n\pi}{l}x \, dx$$

for dcr. $n=1$,

$$n=1, \quad a_n = \frac{2}{l}\int_0^l a \sin\frac{\pi x}{l} \cdot \sin\frac{\pi x}{l} \, dx$$

$$= \frac{2a}{l}\int_0^l \sin^2\frac{\pi x}{l} \, dx \qquad \because \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{2a}{l}\int_0^l \frac{\left[1-\cos\frac{2\pi x}{l}\right]}{2} dx$$

$$= \frac{2a}{l}\times\frac{1}{2}\left[x - \frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right]_0^l$$

$$= \frac{2a}{l}\cdot\frac{1}{2}\left[\left(l - \frac{\sin 2\pi}{\frac{2\pi}{l}}\right) - 0\right]$$

$$= \frac{2a}{l}\cdot\frac{1}{2}\left[l-0\right] = 2a\times\frac{1}{2} = a$$

$$\boxed{a_n = a}$$

$\therefore$ displacement $U(x,t) = y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{cn\pi}{l}t \cdot \sin\frac{n\pi}{l}x$

$$\Rightarrow y(x,t) = a\cos\frac{\pi c t}{l} \cdot \sin\frac{\pi x}{l}$$

ONE DIMENSIONAL HEAT EQUATION

*. Solution of one dimensional heat transfer equation by the method of seperation of variables

Solution:

Equation of heat transfer equation is,

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \quad \text{--- (1)}$$

'U' is the temperature which depends on 't' & 'x'.
by method of seperation of variables, general solution is,

$$U = XT$$

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting all these values in eqn (1), we get

$$(1) \Rightarrow \quad XT' = c^2 X''T \qquad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{X''}{X} = \frac{T'}{T} \cdot \frac{1}{c^2}$$

Using seperation constant, k

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{T'}{T} \cdot \frac{1}{c^2} = k$$

Now $\frac{X''}{X} = k$

$\Rightarrow X'' - Xk = 0$

$(D^2 - k)X = 0$

A.E $\Rightarrow \lambda^2 - k = 0$

$\lambda^2 = k$ — (a)

$\frac{T'}{T}\frac{1}{c^2} = k$

$T' - c^2kT = 0$

$(D - c^2k) = 0$

A.E is $\lambda - c^2k = 0$

$\lambda = c^2k$ — (b)

These two Auxillary equation depends on value of k. Thus there are three cases.

Case I :- when $k=0$

(a) $\Rightarrow \lambda^2 = k$

$\Rightarrow \cancel{\lambda^2} \Rightarrow k = 0$

$\Rightarrow \lambda = 0, 0$

$\therefore X = [c_1 + c_2 x]e^{0x}$

$$\boxed{X = c_1 + c_2 x}$$

(b) $\Rightarrow \lambda = c^2k$

when $k=0$,

$\lambda = c^2k = 0$

ie $\lambda = 0$

soln is, $T = c_3e^{0x} = c_3$

ie $\boxed{T = c_3}$

$\therefore$ general solution is, $U = XT$

ie $\boxed{U = [c_1 + c_2 x]\, c_3.}$ — (A)

Case II :- when k is positive. ie $k = p^2$

(a) $\Rightarrow \lambda^2 = k$, when $k = p^2$

ie $\lambda^2 = p^2$

$\Rightarrow \lambda = \pm p$

$\therefore \boxed{X = c_1e^{px} + c_2e^{-px}}$

(b) $\Rightarrow \lambda = c^2k$, when $k = p^2$

$\lambda = c^2p^2$

$\therefore \boxed{T = c_3e^{c^2p^2t}}$

$\therefore$ soln is $U = XT$

ie $\boxed{U = \left[c_1 e^{px} + c_2 e^{-px}\right]\left[c_3 e^{c^2p^2t}\right]}$ — (B)

Case III :- when k is negative, ie $k = -p^2$

(a) $\Rightarrow \lambda^2 = k$

when $k = -p^2$,

$\lambda^2 = -p^2$

$\Rightarrow \lambda = \pm ip$

$\boxed{X = c_1 \cos px + c_2 \sin px}$

(b) $\Rightarrow \lambda = c^2 k$

when $k = -p^2$,

$\lambda = -c^2 p^2$

$\therefore$ $\boxed{T = c_3 e^{-c^2p^2t}}$

$\therefore$ Soln is $U = XT$

ie $\boxed{U = \left[c_1 \cos px + c_2 \sin px\right]\left[c_3 e^{-c^2p^2t}\right]}$ — (C)

which is the most suitable solution for one dimensional heat equation.

Q) Determine the solution of one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $U(0,t) = 0$ & $U(l,t) = 0$ and initial conditions are, is $U(x,0) = f(x)$ where 'l' be the length of the bar.

Soln:

One dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution of heat equation is

$$U(x,t) = [c_1 \cos px + c_2 \sin px] \, c_3 e^{-c^2p^2t} \quad \text{— (a)}$$

applying 1st boundary conditions,

(1) $U(0,t) = 0$

$$\Rightarrow c_1 \cdot c_3 e^{-p^2c^2t} = 0$$

$$\Rightarrow \boxed{c_1 = 0} \quad \because c_3 \neq 0$$

$$\therefore (a) \Rightarrow [c_2 \sin px] \, c_3 e^{-p^2c^2t} \quad \text{— (b)}$$

applying 2nd boundary condition

(2) $U(l,t) = 0$

$$\therefore (b) \Rightarrow [c_2 \sin pL] \, c_3 e^{-p^2c^2t} = 0$$

$$\Rightarrow \sin pl = 0 \Rightarrow pl = n\pi$$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

$$\therefore U(x,t) = \left[c_2 \sin \frac{n\pi}{l} x\right] c_3 \, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

Combining the constants and adding up all the solutions we get,

$$\therefore \boxed{U(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}}$$

Now applying initial condition.

(3) $U(x,0) = f(x)$.

ie $U(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \; e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$

applying I.C $U(x,0) = f(x)$, we get

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x = f(x)$$, which is half range sine series

where $a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$

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8. A LONG INSULATED ROD WITH ENDS AT ZERO TEMPERATURE

Let us consider a rod of length 'l' with given initial temperature distribution along its axis and ends at constant zero temperature. The lateral surface of the rod is insulated which prevents heat flux in the radial direction and hence the temperature will depend on x-coordinates only. In this case the problem of heat conduction is to solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

under the boundary conditions,

$u(0,t) = 0$ and $u(l,t) = 0$

and the initial condition is,

$u(x,0) = f(x)$.

<u>Note</u>: 1. The boundary conditions with zero end temperature are known as Homogenous boundary conditions.

∴ general solution of one dimensional heat eqn is,

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin \frac{n\pi}{l} x \right] e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}.$$

where $a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$

Q) Determine the solution of onedimensional heat eqn $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the conditions are $u(0,t) = 0$ & $u(l,t) = 0$ and the initial condition is $u(x,0) = x$. ?

<u>Solution</u>:-

one dimensional heat eqn is,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

solution of eq heat eqn is

$$U(x,t) = [c_1 \cos px + c_2 \sin px] c_3 e^{-p^2c^2t}.$$

Boundary condition (BC)

1) $U(0,t) = 0 \Rightarrow c_1 = 0$

2) $U(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$

Initial conditions IC

3) $U(x,0) = f(x) = x.$

applying boundary conditions,

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l}x \; e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}.$$

where $a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi}{l}x \, dx$

$$= \frac{2}{l}\int_0^l x \cdot \sin \frac{n\pi}{l}x \, dx$$

$$\Rightarrow a_n = \frac{2}{l}\left\{\left[x \cdot \frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right]_0^l - \int_0^l 1 \cdot \frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}} \, dx\right.$$

$$= \frac{2}{l}\left\{-\left[\frac{l \cdot \cos n\pi}{\frac{n\pi}{l}}\right] + \frac{l}{n\pi}\int_0^l \cos\frac{n\pi}{l}x \, dx\right.$$

$$= \frac{2}{l}\left\{-\frac{l^2}{n\pi}\cos n\pi + \frac{l}{n\pi}\left[\frac{\sin\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right]_0^l\right.$$

$$= \frac{2}{l}\left\{\frac{-l^2}{n\pi}\cos n\pi + \frac{l^2}{(n\pi)^2}(\sin n\pi - \sin 0)\right\}$$

$$= \frac{2}{l}\left\{-\frac{l^2}{n\pi}\cos n\pi + \frac{l^2}{(n\pi)^2}(0)\right\}$$

$$\Rightarrow a_n = \frac{2}{l}\left\{-\frac{l^2}{n\pi}(-1)^n\right\}$$

$$\boxed{a_n = \frac{-2l}{n\pi}(-1)^n}$$

$\therefore$ Soln is,

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x \; e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

$$\therefore U(x,t) = \sum_{n=1}^{\infty} \frac{-2l}{n\pi}(-1)^n \sin\frac{n\pi}{l}x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

---

(2) Solve the eqn $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ with boundary conditions $U(0,t) = 0$, $U(l,t) = 0$, and initial condition $U(x,0) = 3\sin\frac{n\pi x}{l}$

Soln:

One dimensional heat eqn is $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

Solution of heat eqn is,

$$U(x,t) = [c_1\cos px + c_2\sin px]\, c_3\, e^{-p^2c^2t}$$

~~Applyi~~ BC

1) $U(0,t) = 0 \Rightarrow c_1 = 0$

2) $U(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$

IC

i) $U(x,0) = f(x) = 3\sin\frac{n\pi}{l}x.$

applying Boundary and Initial conditions, we get,

$$U(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin \frac{n\pi}{l} x \right] e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

where $a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$

$$\Rightarrow a_n = \frac{2}{l} \int_0^l 3 \sin \frac{n\pi}{l} x \cdot \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2}{l} \int_0^l 3 \sin^2 \frac{n\pi}{l} x \, dx$$

$$= \frac{6}{l} \int_0^l \frac{\left(1 - \cos \frac{2n\pi}{l} x\right)}{2} dx$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$= \frac{3}{l} \int_0^l \left(1 - \cos \frac{2n\pi}{l} x\right) dx$$

$$= \frac{3}{l} \left[x - \frac{\sin \frac{2n\pi}{l} x}{\frac{2n\pi}{l}} \right]_0^l$$

$$= \frac{3}{l} \left[l - \frac{(\sin 2n\pi - 0)}{\frac{2n\pi}{l}} \right]$$

$$= \frac{3}{l} \left[l - 0 \right] = \underline{\underline{3}}$$

$$\Rightarrow \boxed{a_n = 3}$$

$\therefore$ solution is $U(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$

ie $U(x,t) = \sum_{n=1}^{\infty} 3 \sin \frac{n\pi}{l} x \, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$.

Q) Find the temperature $U(x,t)$ in a slab whose ends $x=0$ & $x=l$ are kept at zero temperature and whose initial temperature $f(x)$ is given by,

$$f(x) = \begin{cases} k & , 0<x<l/2 \\ 0 & , l/2<x<l. \end{cases}$$

soln:

Heat eqn is $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

soln of heat eqn $U(x,t) = [c_1 \cos px + c_2 \sin px] c_3 e^{-c^2p^2t}$

B.C

1) $U(0,t) = 0 \Rightarrow c_1 = 0$

2) $U(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$.

I.C

i) $U(x,0) = f(x) = k$

applying boundary and initial conditions,

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \; e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}.$$

where $a_n = \dfrac{2}{l}\displaystyle\int_0^{l} f(x) \sin \frac{n\pi}{l} x \, dx$

$$= \frac{2}{l}\int_0^{l/2} k \cdot \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2k}{l}\int_0^{l/2} \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2k}{l}\left[\frac{-\cos \frac{n\pi}{l} x}{n\pi/l}\right]_0^{l/2}$$

$$\Rightarrow a_n = \frac{2k}{l}\frac{l}{n\pi}\left[-\left(\cos\frac{n\pi}{l}\frac{l}{2} - \cos 0\right)\right]$$

$$= \frac{2k}{n\pi}\left[-\left(\cos\frac{n\pi}{2} - 1\right)\right]$$

$$\Rightarrow \boxed{a_n = \frac{2k}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]}$$

$\therefore$ soln is,

$$U(x,t) = \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left[1-(-1)^n\right] \sin\frac{n\pi}{l}x \; e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

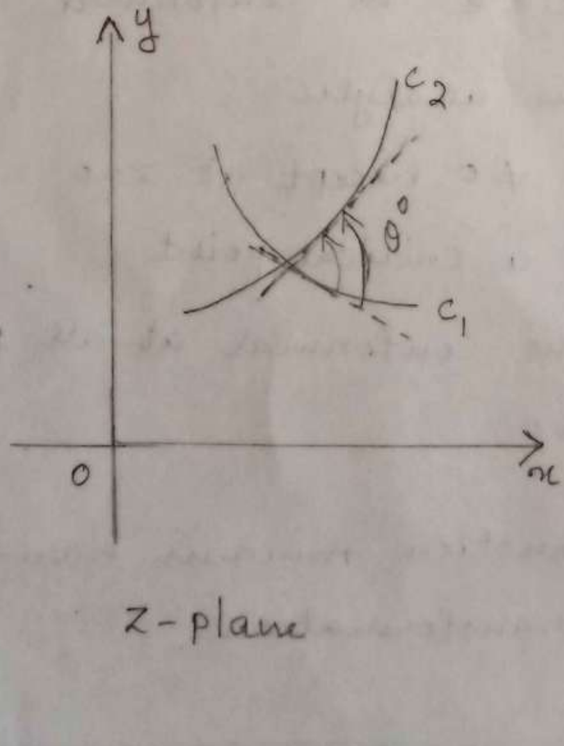
CONFORMAL MAPPING

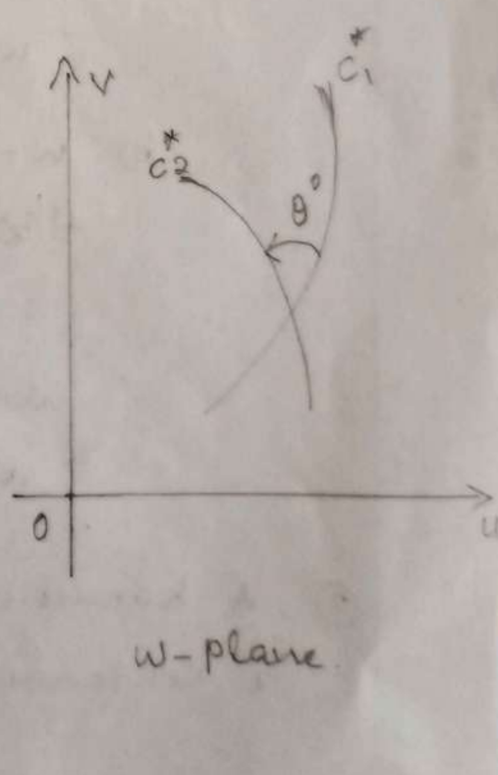
The complex function $w = f(z)$

ie $\quad u+iv = f(x+iy)$

involves 4 variables, 2 independent variables x and y and 2 dependent variables u and v. A 4-dimensional region is required to represent it graphically. As it is not possible, we choose two complex planes z-plane and w-plane.

In the z-plane we plot $z = x+iy$ and in the w-plane we plot $w = u+iv$. So the function $w = f(z)$ defines a correspondance between points of these two planes.





A transformation which preserves magnitude and sense of the angle between every pair of curves in some domain in the z-plane as its image in the w-plane is called Conformal mapping.

A transformation which preserves angles in magnitude alone is called isogonal mapping.

<u>Note</u>:

① If $w = f(z)$ is analytic then it is Conformal at each points of its domain, provided $f'(z) \neq 0$. The points at which $f'(z) = 0$ are called critical points.

Eg: a) $w = e^z$ is analytic

$f'(z) = e^z \neq 0$ for all z.

$\therefore w = f(z) = e^z$ is Conformal.

b) $w = z^2$ is analytic.

$f'(z) = 2z \neq 0$ except at $z = 0$

$z = 0$ is a critical point

$\therefore w = z^2$ is Conformal at all points except at $z = 0$.

② A harmonic function remains harmonic under a Conformal transformation.

Q) Discuss the mapping or transformation of $w = z^2$

<u>Soln:</u>

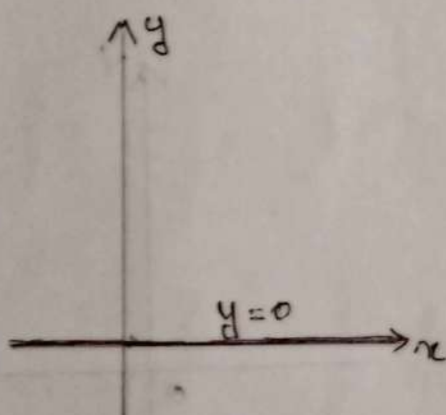
$$w = z^2$$

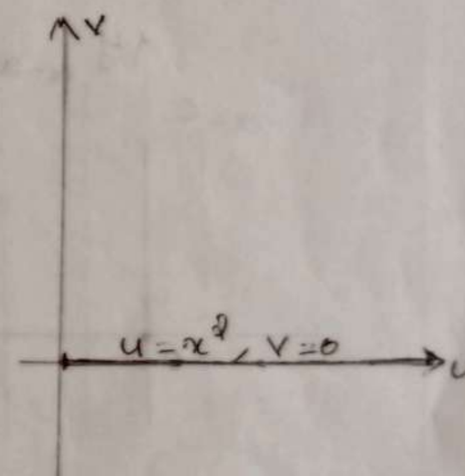
$$u + iv = (x+iy)^2 = x^2 - y^2 + i\,2xy.$$

$$\Rightarrow u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

<u>Case 1:</u>

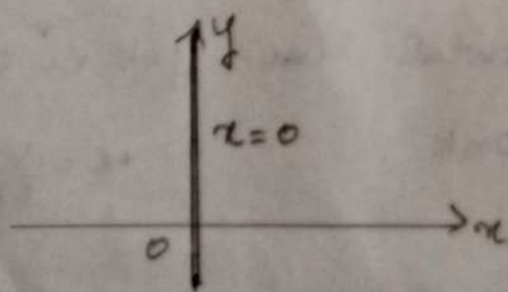
Consider the real axis $y=0$ in the z-plane. then $u = x^2$, $v = 0$; which is the positive real axis in the w-plane.



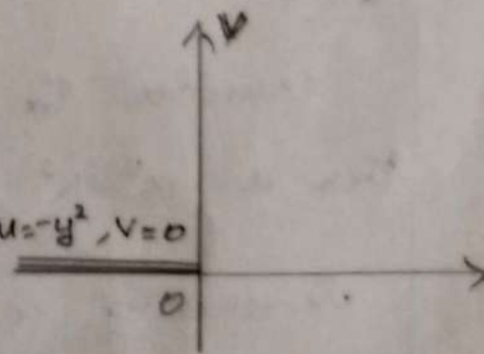


<u>Case 2:-</u>

Consider the imaginary axis $x=0$ in the z-plane then $u = -y^2$, $v=0$; which is the negative real axis in the w-plane.



Z-plane

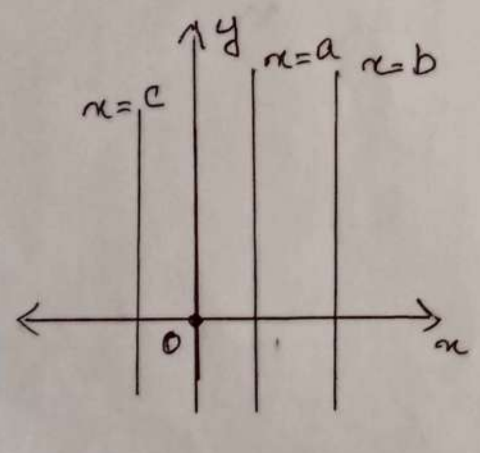


w-plane.

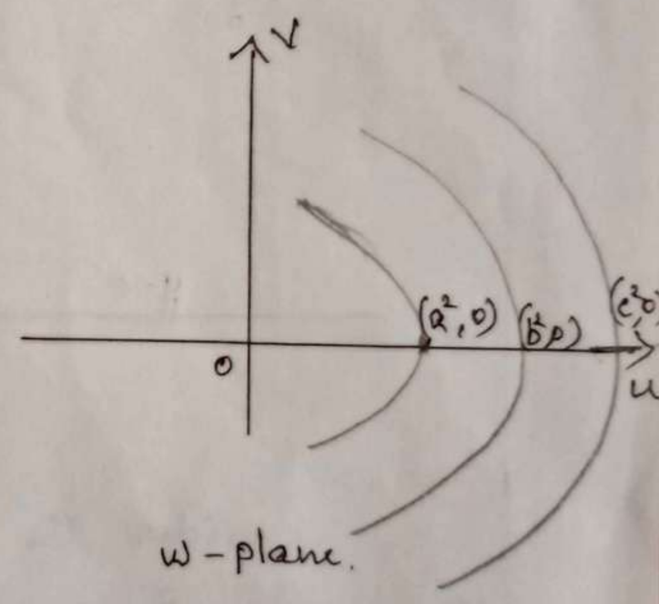
Case 3:- Consider the vertical line $x = c$ in the z-plane then $u = c^2 - y^2$; $v = 2cy$.

eliminating y,
$$u = c^2 - y^2$$
$$u = c^2 - \left(\frac{v}{2c}\right)^2 \qquad \therefore\ y = \frac{v}{2c}$$
$$\frac{v^2}{4c^2} = c^2 - u$$
$$v^2 = 4c^2(c^2 - u)$$
$$(v-0)^2 = -4c^2(u - c^2),$$ which is a left handed parabola with vertex at $(x_1, y_1) = (c^2, 0)$.

$\boxed{(y-y_1)^2 = 4a(x-x_1)}$ Vertex at $(x_1, y_1) \Rightarrow$ parabola



Z-plane.



w-plane.

Case 4:

Consider the horizontal line $y = k$ in the z-plane.

then $u = x^2 - k^2$, $v = 2xk$ $\qquad \therefore\ x = \frac{v}{2k}$

eliminating x,
$$u = x^2 - k^2$$
$$\Rightarrow u = x^2 - \left(\frac{v}{2k}\right)^2$$

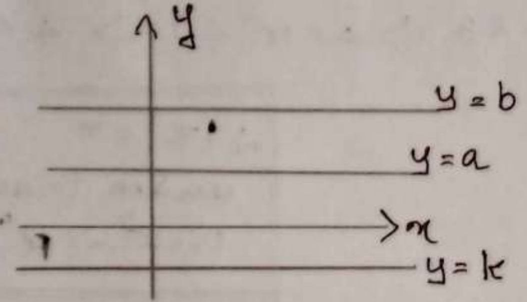
$$\Rightarrow \quad u = \left(\frac{v}{2k}\right)^2 - k^2$$

$$\frac{v^2}{4k^2} = u + k^2$$

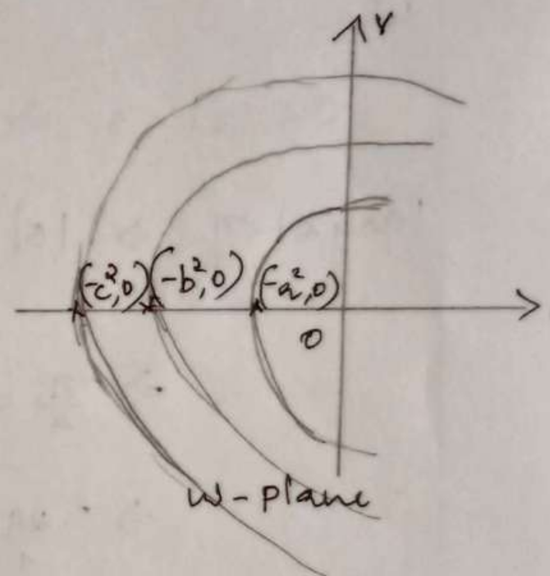
$$v^2 = 4k^2(u+k^2)$$

$$(v-0)^2 = 4k^2(u - -k^2)$$

which is a right handed parabola with vertex at $(-k^2, 0)$



z-plane



Also $f'(z) = 2z \neq 0$ except at $z=0$

Hence the transformation $w = z^2$ is conformal at all points except at $z = 0$.

REMARK

Using polar forms $z = re^{i\theta}$ in the z-plane, $w = Re^{i\phi}$ in the w-plane

$$w = z^2 \Rightarrow Re^{i\phi} = (re^{i\theta})^2$$

$$\Rightarrow Re^{i\phi} = r^2 e^{i2\theta}$$

$$\Rightarrow R = r^2 \text{ and } \phi = 2\theta$$

Hence the circle having radius r_0 is mapped to circle having radius r_0^2 and θ_0 mapped to $2\theta_0$

Q) Find the image of the region $2 < |z| < 3$; $|\arg z| < \frac{\pi}{4}$ under the map $w = z^2$

<u>Soln</u>:-

$$2 < |z| < 3 \Rightarrow 2 < r < 3 \Rightarrow 4 < r^2 < 9 \Rightarrow 4 < R < 9$$

$$|\arg z| < \frac{\pi}{4} \Rightarrow |\theta| < \frac{\pi}{4}$$

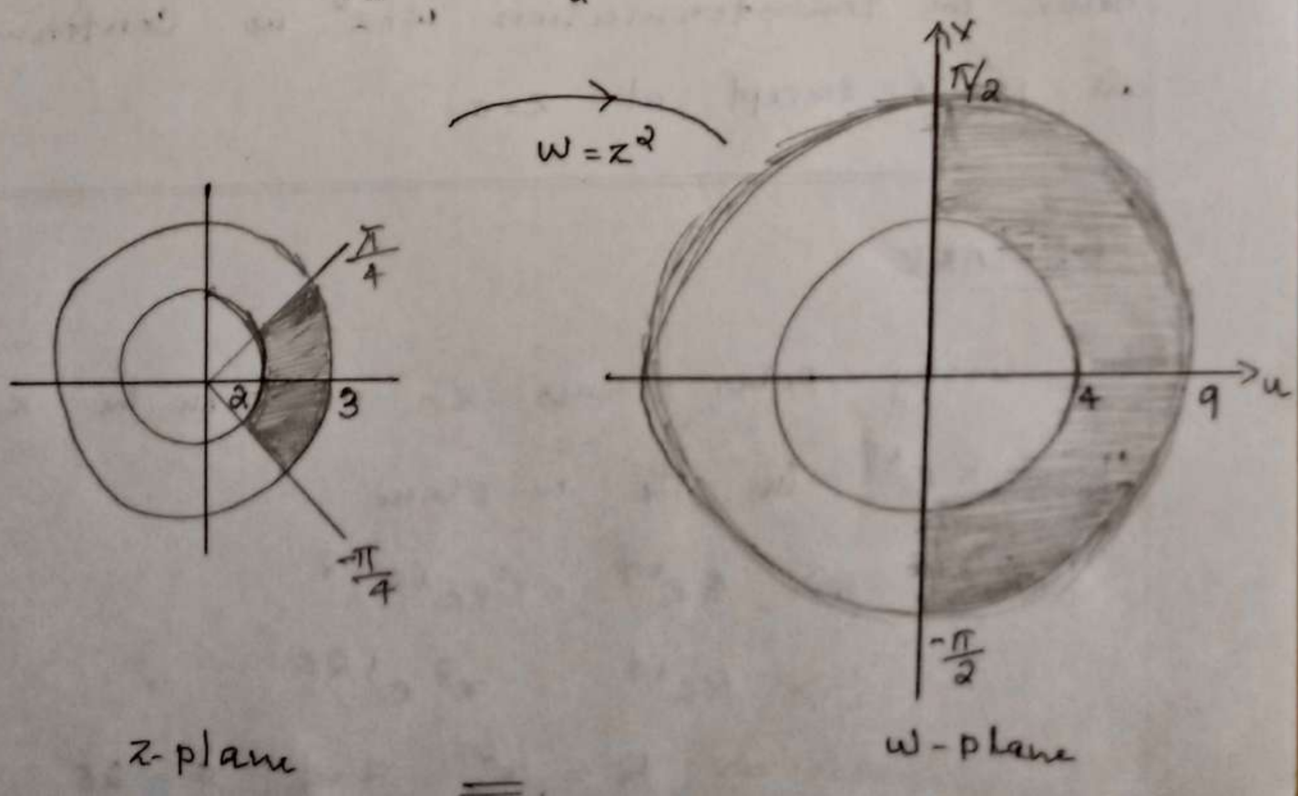
$$\Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$\Rightarrow -\frac{2\pi}{4} < 2\theta < \frac{2\pi}{4}$$

$$\Rightarrow -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

∵ $|z| = r$ under transformation $w = z^2 \Rightarrow R = r^2$ & $\phi = 2\theta$

∴ $\arg z = \theta$ under $w = z^2$, $\phi = 2\theta$.



MAPPING OF $w = e^z$

Q. Conformal mapping of $w = e^z$.

Soln:

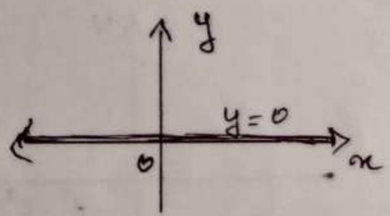
$$w = e^z \Rightarrow u + iv = e^{x+iy} = e^x \cdot e^{iy}$$

$$\Rightarrow u + iv = e^x[\cos y + i \sin y]$$

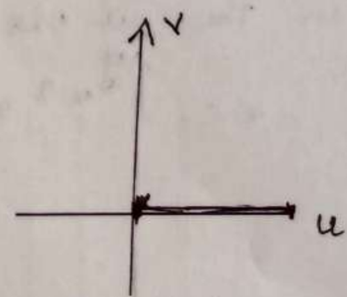
$$\Rightarrow u = e^x \cos y \,, \quad v = e^x \sin y.$$

Case 1:

Consider the real axis $y=0$ in the z-plane then $u = e^x$; $v = 0$ which is positive real axis in the w-plane.



z-plane



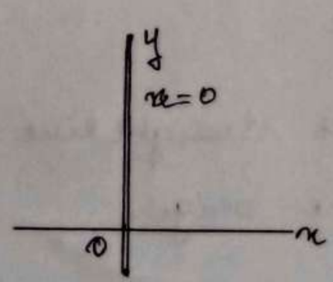
w-plane.

Case 2 :-

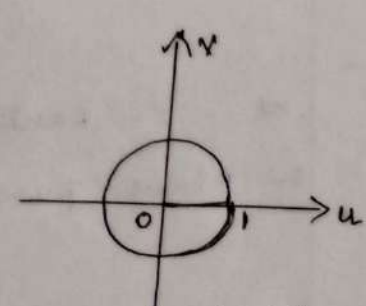
Consider the imaginary axis $x=0$ in the z-plane then $u = \cos y$; $v = \sin y$.

Eliminating y, $\cos^2 y + \sin^2 y = u^2 + v^2$

ie $u^2 + v^2 = 1$, which is a circle in the w-plane with centre at $(0,0)$ & radius 1.



Z-plane



w-plane

Case 3:

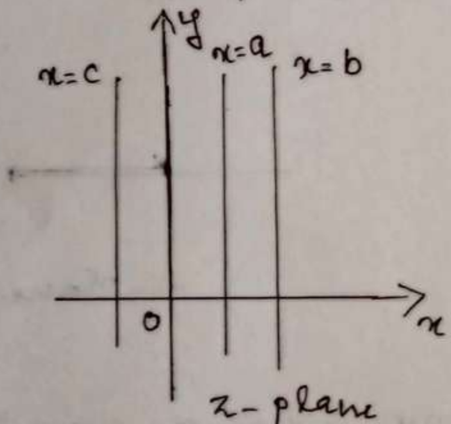
Consider the vertical line $x = c$ in the z-plane, then $u = e^c \cos y$; $v = e^c \sin y$

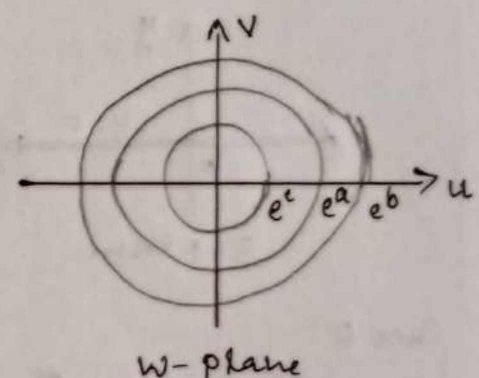
Eliminating y,

$$\cos^2 y + \sin^2 y = \left(\frac{u}{e^c}\right)^2 + \left(\frac{v}{e^c}\right)^2$$

ie $$1 = \frac{u^2}{(e^c)^2} + \frac{v^2}{(e^c)^2}$$

ie $u^2 + v^2 = (e^c)^2$ which is a circle in the w-plane with centre at $(0,0)$ & radius e^c.





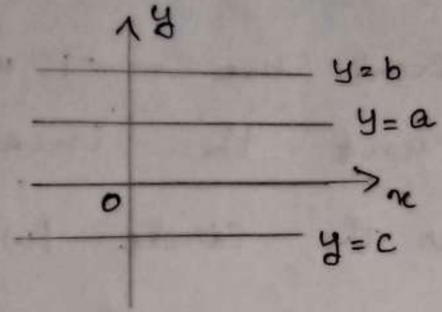
Case 4:-

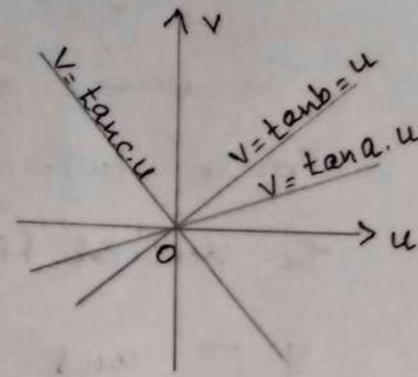
Consider the horizontal lines $y = k$ in the z-plane then $u = e^x \cos k$; $v = e^x \sin k$

Eliminating x,

$$\frac{v}{u} = \tan k$$

ie $v = \tan k \cdot u$; which is a straight line in the w-plane passing through the origin





Q) Find the image of the region $-\log 2 \leq x \leq \log 4$ under the mapping $\omega = e^z$.

Soln:

$$W = e^z$$

$$-\log 2 \leq x \leq \log 4$$

Image of the vertical line $x = -\ln 2$ is the circle

$$|w| = e^{-\ln 2}$$

$$\Rightarrow |w| = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$$

Similarly, the image of the line $x = \ln 4$ is the circle $|w| = e^{\ln 4} \Rightarrow |w| = 4$.

Hence the region bounded by the lines $|w| = \frac{1}{2}$ and $|w| = 4$.

Q) Find the image of the region $-1 \leq x \leq 2, -\pi \leq y \leq \pi$ under the mapping $\omega = e^z$

Soln:

$$-1 \leq x \leq 2 \quad \Rightarrow x = -1, \; x = 2$$

$$-\pi \leq y \leq \pi \quad \Rightarrow y = -\pi, \; y = \pi.$$

The image of the vertical line $x = -1$ is the Circle $|w| = e^{-1}$ ~~is~~ ~~the~~ and the image of the vertical line $x = 2$ is the circle $|w| = e^{2}$.

$y = -\pi$ and $y = \pi$ are mapped on the rays $\arg w = -\pi$ and $\arg w = \pi$ respectively.

Q) Mapping of $w = \frac{1}{z}$

Soln:-

$$u + iv = \frac{1}{x+iy}$$

$$\Rightarrow u + iv = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x - iy}{x^2 + y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

Q) Find the image of the following regions under the mapping $w = \frac{1}{z}$.

1) $\frac{1}{4} < y < \frac{1}{2}$

2) $0 < y < \frac{1}{2}$

3) $|z - 2i| = 2$.

Soln:

(1) $\frac{1}{4} < y < \frac{1}{2}$.

$\frac{1}{4} < y$ and $y < \frac{1}{2}$.

Let $w = \frac{1}{z} = u + iv$

Then $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$

$\frac{1}{4} < y \Rightarrow \frac{1}{4} < \frac{-v}{u^2+v^2}$

$\Rightarrow u^2+v^2 < -4v$

$\Rightarrow u^2+v^2+4v < 0$ which is the interior part of the circle.

Now $y < \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2 > -2v$

$\Rightarrow u^2+v^2+2v > 0$ which is the exterior part of the circle.

$\therefore$ The image of the region $\frac{1}{4} < y < \frac{1}{2}$ is mapped in to the region between the circles $u^2+v^2+4v = 0$ and $u^2+v^2+2v = 0$.

(2) $0 < y < \frac{1}{2}$

Let $w = \frac{1}{z} = u + iv$

$\Rightarrow x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$0<y \Rightarrow 0< \frac{-v}{u^2+v^2}$

$\Rightarrow -v>0 \Rightarrow v>0$ ∴ The image of the $0<y$ is $v>0$, which is the upper half plane

Now $y<\frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2}<\frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v>0$, which is the exterior part of the circle.

∴ $0<y<\frac{1}{2}$, ⇒ Images of $0<y<\frac{1}{2}$ is mapped onto $v>0$ and $\underline{\underline{u^2+v^2+2v>0}}$

③ $|z-2i|=2$

Let $w=\frac{1}{z}=u+iv$

$\Rightarrow x=\frac{u}{u^2+v^2}$ and $y=\frac{-v}{u^2+v^2}$

Now $|z-2i|=2 \Rightarrow |x+iy-2i|=2$

$\Rightarrow |x+i(y-2)|=2$

$\Rightarrow \sqrt{x^2+(y-2)^2}=2$ ∴ $|x+iy|=\sqrt{x^2+y^2}$

Squaring both sides,

$\Rightarrow x^2+(y-2)^2=4$

$\Rightarrow x^2+y^2-4y+4=4$

$\Rightarrow x^2+y^2-4y=0$

Substituting for x^2 and y^2, we get

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}-4\frac{-v}{u^2+v^2}=0$$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2}+\frac{4v}{(u^2+v^2)}=0$$

$$\Rightarrow u^2+v^2+4v(u^2+v^2)=0$$

$$\Rightarrow (u^2+v^2)(1+4v)=0$$

$\Rightarrow 1+4v=0$, which is straight line.

Thus the image of the circle $|z-2i|=2$ is the straight line $1+4v=0$.

Q). MAPPING OF $w = \sin z$

Soln:

$w=\sin z \Rightarrow u+iv=\sin(x+iy)$

$\Rightarrow u+iv=\sin x\cos iy+\cos x\sin iy$

$\Rightarrow u+iv=\sin x\cosh y+i\cos x\sinh y$

$\Rightarrow u=\sin x\cosh y$, $v=\cos x\sinh y$.

$\cos iy=\cosh y$ $\sin iy=i\sinh y$
$\cos 0=1$ $\sin 0=0$

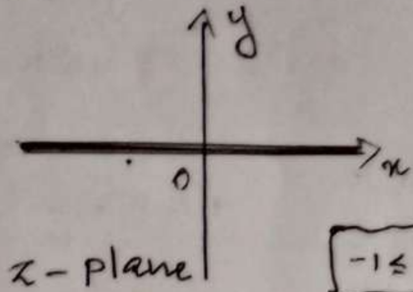
Case 1:

Consider the real axis $y=0$ in the z-plane,

$u=\sin x\cos 0$

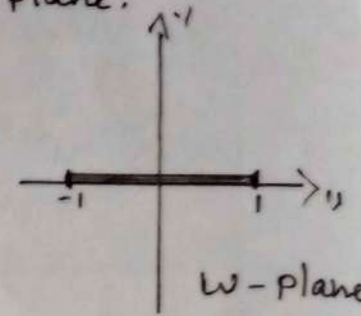
$v=\cos x\sin 0$

$u = \sin x$, $v = 0$ which is the line segment from -1 to 1 on the real axis of the w-plane.



z-plane

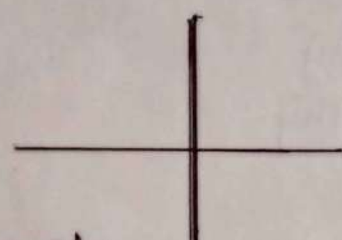
$\boxed{-1 \leq \sin x \leq 1}$



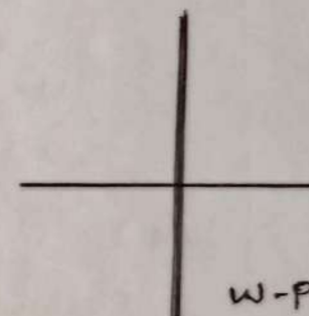
w-plane

Case II

Consider the imaginary axis $x=0$ in the z-plane, then $u=0$; $v = \sinh y$ which is the imaginary axis of the w-plane.



z-plane



w-plane

$\boxed{-\infty \leq \sinh x \leq \infty}$

Case III

Consider the vertical line $x=c$ in the z-plane, then $u = \sin c \cosh y$; $v = \cos c \sinh y$

eliminating y,

$$\cosh^2 y - \sinh^2 y = \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c}$$

ie $$1 = \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c}.$$ since $\cosh^2\theta - \sinh^2\theta = 1$

$\Rightarrow \dfrac{u^2}{\sin^2 c} - \dfrac{v^2}{\cos^2 c} = 1$; which is a hyperbola with foci $(\pm 1, 0)$.

Case IV

Consider the horizontal line $y=k$ in the z-plane, then $u = \sin x \cosh k$, $v = \cos x \sinh k$

Eliminating α,

$$\sin^2\alpha + \cos^2\alpha = \frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k}$$

$\Rightarrow \dfrac{u^2}{\cosh^2 k} + \dfrac{v^2}{\sinh^2 k} = 1$ which is an ellipse in the w-plane with focci $(\pm 1, 0)$.

Fixed points or Invariant points.

Fixed points of a mapping $w = f(z)$ are points that are mapped on to themselves. The fixed-points are obtained by putting $w = z$.

Q) Find the fixed points of the mapping

$$w = \frac{3z-5i}{iz-1}.$$

Soln: put $w = z$

$$\Rightarrow z = \frac{3z-5i}{iz-1}$$

$$iz^2 - z = 3z - 5i$$

$$iz^2 - 4z + 5i = 0$$

$$\Rightarrow z = \frac{4 \pm \sqrt{16-20i^2}}{2i} = \frac{4 \pm \sqrt{36}}{2i} = \frac{4 \pm 6}{2i}$$

$$\Rightarrow \frac{10}{2i}, \frac{-2}{2i}$$

$\Rightarrow -5i \ , \ i$

$\therefore$ The fixed points are $z = -5i$ and i.

H.W

1) Find the fixed points of the transformations

$$w = \frac{5-4z}{4z-2}$$

2) Find the image of the following region under the mapping $w = \frac{1}{z}$.

$|z-3| = 5$.

MODULE - IV

COMPLEX INTEGRATION

* **Complex Line Integral**

Let $f(z)$ be a continous function of z, defined at all points of a curve c having end points A and B, then the complex integral of $f(z)$ along c is $\int_c f(z)\,dz$.

'c' is called the path of integration. If c is a closed curve the integral is called contour integral and is denoted by $\oint_c f(z)\,dz$.

* **FIRST EVALUATION METHOD**

Let $f(z)$ be analytic in a simple connected domain D and $F(z)$ be an analytic function such that $F'(z) = f(z)$. then,

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

Eg: $\int_0^{1+i} z^2\,dz = \left.\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$

$$= \frac{1+3i-3-i}{3} = \frac{-2+2i}{3}$$

* ## SECOND EVALUATION METHOD

The first evaluation method is restricted to analytic functions. But the second evaluation method can be applied to any continous complex functions.

$z = x+iy$
$dz = dx + i\,dy$

ie $$\int_C f(z)\,dz = \int_{t=a}^{b} f(z(t))\,\frac{dz}{dt}\,dt$$ $$\boxed{= \int f(x+iy)(dx+i\,dy)}$$

Q)1. Evaluate $\int_C \bar{z}\,dz$ where C is given by $x = 3t$, $y = t^2-1$, $-1 \le t \le 4$.

Soln:

$$\int_C \bar{z}\,dz = \int (x-iy)(dx+i\,dy) \quad \because dz = dx+i\,dy$$

$$= \int_{t=-1}^{4} [3t - i(t^2-1)]\,[3\,dt + i\,2t\,dt]$$

$$= \int_{-1}^{4} (3t - it^2 + i)(3+2it)\,dt$$

$$= \int_{-1}^{4} (9t + 6it^2 - 3it^2 + 2t^3 + 3i - 2t)\,dt$$

$$= \int_{-1}^{4} (7t + 3it^2 + 2t^3 + 3i)\,dt$$

$$= \left[\frac{7t^2}{2} + 3i\frac{t^3}{3} + 2\frac{t^4}{4} + 3it\right]_{-1}^{4}$$

$$= (56 + 64i + 128 + 12i) - (7/2 - i + \tfrac{1}{2} - 3i)$$

$$= \underline{\underline{180 + 80i}}$$

2. Evaluate $\int_0^{1+2i} f(z)\,dz$ where $f(z) = Re(z)$

i) Along the straight line from 0 to $1+2i$

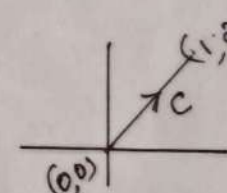
ii) Along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+2i$

<u>Soln:</u>

$$f(z) = Re(z) = x.$$

i) Along the straight line from 0 to $1+2i$

$$\int_C f(z)\,dz = \int_C x\,(dx + i\,dy)$$



Along C; $y = 2x$

$\Rightarrow \quad dy = 2\,dx;$

$\Rightarrow x$ from 0 to 1

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$(x_1, y_1) = (0,0)$$

$$(x_2, y_2) = (1,2)$$

$$\therefore m = \frac{2-0}{1-0} = 2$$

$$\therefore \int_C f(z)\,dz = \int_{x=0}^{1} x\,(dx + i\,2\,dx)$$

$$= (1+2i)\int_0^1 x\,dx$$

$$= (1+2i)\left[\frac{x^2}{2}\right]_0^1$$

$$= (1+2i)\,\frac{1}{2} = \frac{1}{2} + i$$

$$\Rightarrow \int_C f(z)\,dz = \underline{\underline{\frac{1}{2} + i}}$$

※ $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

ii) Along the real axis from $z=0$ to $z=1$ and parallel to Imaginary axis from $z=1$ to $z=1+2i$

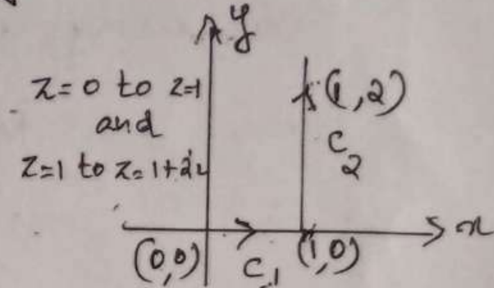
Soln: $z \Rightarrow 0 \Rightarrow (0,0)$ $\qquad z = x+iy$

$z = 1 \Rightarrow (1,0)$

and parallel to imaginary axis from $z=1$ to $z=1+2i$

$\Rightarrow z=1 \Rightarrow (1,0)$

$z=1+2i \Rightarrow (1,2)$



$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$C_1 : \Rightarrow (0,0)$ to $(1,0)$

$C_2 : \Rightarrow (1,0)$ to $(1,2)$

$\therefore C = C_1 + C_2$.

$$= \int_{C_1} x\,(dx+i\,dy) + \int_{C_2} x\,(dx+i\,dy)$$

Along C_1, $y = mx = \frac{0-0}{1-0}x = 0 \Rightarrow y=0$.

$\Rightarrow dy = 0$; x from 0 to 1.

Along C_2, $(1,0)$ to $(1,2)$

$\Rightarrow x=1 \Rightarrow dx = 0$

$\Rightarrow y$; 0 to 2.

$$\therefore \int_C f(z)\,dz = \int_{x=0}^{1} x\,dx + \int_{y=0}^{2} 1.\,i\,dy$$

$$= \left[\frac{x^2}{2}\right]_0^1 + i\,[y]_0^2$$

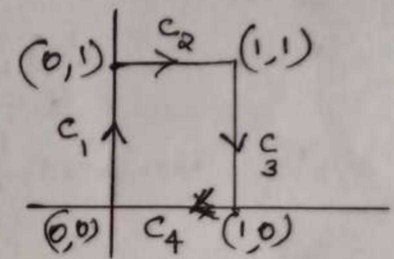
$$= \frac{1}{2} + 2i$$

Q) Evaluate $\oint_c Re(z^2)\,dz$ over the boundary c of the square with vertices $0, i, 1+i, 1$ clockwise.

Soln:

$z^2 = (x^2 - y^2) + i\,2xy$

$Re(z^2) = x^2 - y^2$



Along c_1, $x = 0 \Rightarrow dx = 0$; y : 0 to 1.

Along c_2, $y = 1 \Rightarrow dy = 0$; x : 0 to 1

Along c_3, $x = 1 \Rightarrow dx = 0$; y : 1 to 0

Along c_4, $y = 0 \Rightarrow dy = 0$; x : 1 to 0.

$$\therefore \int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \int_{c_4} f(z)\,dz$$

$$\Rightarrow \int_{c_1}(x^2-y^2)(dx + i\,dy) + \int_{c_2}(x^2-y^2)(dx+i\,dy) + \int_{c_3}(x^2-y^2)(dx+i\,dy) + \int_{c_4}(x^2-y^2)(dx+i\,dy)$$

$$= \int_{y=0}^{1} -y^2\, i\,dy + \int_{x=0}^{1}(x^2-1)\,dx + \int_{y=1}^{0}(1-y^2)\,i\,dy + \int_{x=1}^{0} x^2\,dx$$

$$= i\left[\frac{-y^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - x\right]_0^1 + i\left[y - \frac{y^3}{3}\right]_1^0 + \left[\frac{x^3}{3}\right]_1^0$$

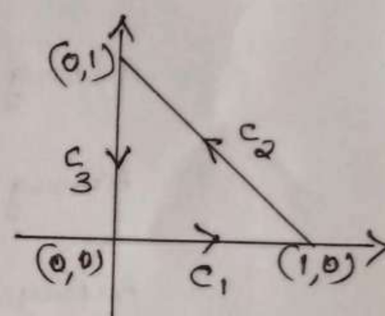
$$\Rightarrow \left(\frac{-i}{3}\right) + \left(\frac{1}{3} - 1\right) + i\left(\frac{-2}{3}\right) + \frac{-1}{3}$$

$$\Rightarrow \quad = \quad \frac{-i}{3} - \frac{2}{3} - \frac{2}{3}i - \frac{1}{3} = -1-i$$

$$\therefore \oint_C f(z)\, dz = -1-i$$

Q) Evaluate $\int_C \mathrm{Im}(z^2)\, dz$ where C is the triangle with vertices $0, 1, i$ counter clock wise.

<u>Soln:</u>



$$z^2 = x^2 - y^2 + 2ixy$$

$$\therefore \mathrm{Im}(z^2) = 2xy.$$

Along c_1, $y=0$, $dy=0$, x: 0 to 1.

Along c_2, $y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$\Rightarrow y = -x+1 \Rightarrow dy = -dx, \; x: 1 \text{ to } 0$$

Along c_3, $x=0 \Rightarrow dx=0$, y: 1 to 0.

$$\therefore \int_C \mathrm{Im}(z^2)\, dz = \int_{c_1} 2xy\,(dx + i\,dy) + \int_{c_2} 2xy\,(dx + i\,dy) + \int_{c_3} 2xy\,(dx + i\,dy)$$

$$\Rightarrow \quad = \int_{x=0}^{1} 0 + \int_{x=1}^{0} 2x(-x+1)(dx - i\,dx) + \int_{y=1}^{0} 0.$$

$$= \int_{x=1}^{0} 2x(-x+1)(1-i)\,dx$$

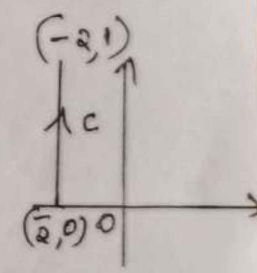
$$= (1-i)\int_{x=1}^{0} (-2x^2 + 2x)\,dx$$

$$= (1-i)\left[-\frac{2x^3}{3} + 2\frac{x^2}{2}\right]_1^0$$

$$= (1-i)\left[\frac{2}{3} - 1\right] = (1-i)\frac{-1}{3}$$

$$= \frac{-(1-i)}{3}$$

Q) Show that $\int_C (2+z)^2\,dz = \frac{-i}{3}$ where C is any path connecting the points -2 & $-2+i$.



-2 to $-2+i$

$\Rightarrow (-2,0)$ to $(-2,1)$

<u>Soln</u>:

$$\int_C (2+z)^2\,dz = \int_C (2+x+iy)^2(dx+i\,dy)$$

Along C, $x = -2 \Rightarrow dx = 0$; $y = 0$ to 1

$$\therefore \int_C (2+z)^2\,dz = \int_{y=0}^{1} (2-2+iy)^2 (i\,dy)$$

$$= \int_0^1 (iy)^2\, i\,dy$$

$$= -i\int_0^1 y^2\,dy$$

$$= -i\left[\frac{y^3}{3}\right]_0^1 = \frac{-i}{3}$$

$$\therefore \int_C (2+z)^2\,dz = \frac{-i}{3} \;//.$$

*

CAUCHY'S INTEGRAL THEOREM

If $f(z)$ is analytic within or on a closed path c then $\oint_c f(z)\,dz = 0$

Eg:- ① $\oint_c \frac{1}{z^2+1}\,dz$; $c : |z| = \frac{1}{2}$

$z^2+1=0$
$z^2=-1$
$\Rightarrow z = \pm\sqrt{-1}$
$\Rightarrow z = \pm i$
$(z = i, -i)$

Here $f(z)$ is not analytic at $z = \pm i$

at $z=i$, $|+i| = |0+1i| = 1 > \frac{1}{2}$

at $z=-i$, $|-i| = |0-1i| = 1 > \frac{1}{2}$

$\therefore$ The points $z = \pm i$ lies outside c.

$\therefore f(z)$ is analytic inside c.

By Cauchy's Integral theorem, $\oint_c f(z)\,dz = 0$

$$\Rightarrow \oint_c \frac{1}{z^2+1}\,dz = 0$$

---

② $\oint_c \frac{1}{z-3}\,dz$; $|z|=1$

Here $f(z)$ is not analytic at $z=3$

$|3| = 3 > 1$, lies outside c; $|z|=1$.

$\therefore f(z)$ is analytic inside c.

$\therefore$ By cauchy's integral theorem, $\oint_c f(z)\,dz = 0$

$$\Rightarrow \oint_c \frac{1}{z-3}\,dz = 0$$

CAUCHY'S INTEGRAL FORMULA (CIF)

Let $f(z)$ be analytic within or on a closed contour c, then $\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$, z_0 is a point within c, the integration is being taken in the counter clockwise.

Also $\oint_c \frac{f(z)}{(z-z_0)^2}\,dz = \frac{2\pi i}{1!} f'(z_0)$

$$\oint_c \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$

In general, $\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$.

Q) Evaluate $\oint_c \frac{\cos \pi z}{z-2}\,dz$; $c : |z| = 3$

<u>Soln</u>:-

Here $\frac{\cos \pi z}{z-2}$ is not analytic at $z = 2$.

But when $z = 2$, $|z| = 2 < 3$

$\therefore$ $z = 2$ lies within c.

By cauchy's Integral formula,

$$\oint_c \frac{\cos \pi z}{z-2}\,dz = 2\pi i\, f(2), \text{ where,}$$

$$= 2\pi i \times 1 = \underline{2\pi i}$$

$f(z) = \cos \pi z$

$f(2) = \cos 2\pi = 1$

Q) Evaluate $\oint_C \frac{e^z}{z+1} dz$ where C is

i) $|z| = 2$

ii) $|z| = \frac{1}{2}$

(iii) $|z+1| = \frac{1}{2}$.

Soln:

$f(z) = \frac{e^z}{z+1}$ is not analytic at $z = -1$.

i) $|z| = 2$

when $z = -1$, $|z| = |-1| = 1 < 2$.

$\therefore z = -1$ lies inside C.

$\therefore z_0 = -1$

By Cauchy's Integral formula, (CIF),

$$\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i \times f(z_0)$$

$$\Rightarrow \oint_C \frac{e^z}{(z-^-1)} dz = 2\pi i \times f(-1)$$, where $f(z) = e^z$, $f(z_0) = f(-1) = e^{-1}$

$$= 2\pi i \times e^{-1}$$

$$= \frac{2\pi i}{e}$$

---

ii) $|z| = \frac{1}{2}$

Soln: C is $|z| = \frac{1}{2}$

$z_0 = -1$.

when $z = -1$, $|z| = |-1| = 1 > \frac{1}{2}$

$\therefore z = -1$ lies outside C.

ie, apply Cauchy's Integral thm.

$\therefore$ By cauchy's Integral theorem,

$$\oint_C f(z)\, dz = 0$$

$$\Rightarrow \oint_C \frac{e^z}{z+1}\, dz = 0.$$

iii) $|z+1| = \frac{1}{2}$

when $z=-1$, $|z+1| = |0| = 0 < \frac{1}{2}$

$\therefore z=-1$ lies inside c.

$\therefore$ By CIF, $\oint_C \frac{e^z}{z+1}\, dz = 2\pi i \times f(-1)$

where $f(z) = e^z$

$f(z_0) = f(-1) = e^{-1}$

$$= 2\pi i \times e^{-1} = \frac{2\pi i}{e}.$$

Q) Calculate $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z-2)}$; c is $|z| = 3$.

Soln:

Here $(z+1)(z-2) = 0$; c; $|z| = 3$

$\Rightarrow z = -1, 2$ are singular points.

when $z=-1$, $|z| = |-1| = 1 < 3 \Rightarrow z=-1$ lies inside c.

when $z=2$, $|z| = |2| = 2 < 3 \Rightarrow z=2$ lies inside c.

$\therefore z=-1, 2$ lies inside c.

$\therefore z_0 = -1, 2$

$$\boxed{CIF \Rightarrow \oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i \times f(z_0)}$$

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

$\therefore 1 = A(z-2) + B(z+1)$

put $z=2 \Rightarrow 3B = 1 \Rightarrow \boxed{B = \frac{1}{3}}$

put $z = -1 \Rightarrow -3A = 1 \Rightarrow \boxed{A = \frac{-1}{3}}$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{\frac{-1}{3}}{z+1} + \frac{\frac{1}{3}}{z-2}$$

$$\therefore \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z-2)} = \left(-\tfrac{1}{3}\right)\frac{\sin\pi z^2 + \cos\pi z^2}{z+1} + \left(\tfrac{1}{3}\right)\frac{\sin\pi z^2 + \cos\pi z^2}{z-2}.$$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z-2)} = \frac{-1}{3}\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z+1} + \frac{1}{3}\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}$$

$$\Rightarrow \quad = -\frac{1}{3} \times 2\pi i \times f(-1) + \frac{1}{3} 2\pi i \times f(2)$$

$$\Rightarrow \quad = -\frac{1}{3} \times 2\pi i \times -1 + \frac{1}{3} \times 2\pi i \times 1$$

$$= \frac{2\pi i}{3} + \frac{2\pi i}{3}$$

$$= \frac{4}{3}\pi i$$

where $f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(-1) = \sin\pi + \cos\pi =$
$= 0 + -1$
$\Rightarrow f(-1) = -1$

$f(2) = \sin 4\pi + \cos 4\pi$
$= 0 + 1$
$\Rightarrow f(2) = 1$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z+2)} = \frac{4}{3}\pi i$$

Q) Evaluate $\oint_c \frac{e^z}{(z+1)^3} dz$; c is $|z+1|=1$

<u>Soln</u>:

$z=-1$ is a singular point.

when $z=-1$, ~~$|z|=|-1|=1<$~~ $|z+1|=|-1+1|=0<1$

$\therefore z_0=-1$ lies inside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{n}(z_0)$

$$\Rightarrow \oint_c \frac{e^z}{(z--1)^3} dz = \oint_c \frac{e^z}{(z--1)^{2+1}} dz$$

$$= \frac{2\pi i}{2!} \cdot f''(z_0)$$

$$= \frac{2\pi i}{2!} \times e^{-1}$$

$$\therefore \oint_c \frac{e^z}{(z+1)^3} dz = \frac{\pi i}{e}$$

where $f(z)=e^z$

~~$f(z_0)=f(-1)=e^{-1}$~~

$f'(z)=e^z$

$f''(z)=e^z$

$f''(z_0)=f''(-1)=e^{-1}$

Q) Evaluate $\oint \frac{\sin 2z}{z^4} dz$; c is $|z|=1$

<u>Soln</u>:

$z=0$ is a singular point.

when $z=0$, $|z|=0<1$

$\therefore z=z_0=0$ lies inside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} \times f^{n}(z_0)$.

ie $\oint_C \frac{\sin 2z}{z^4}\,dz = \oint_C \frac{\sin 2z}{z^{3+1}}\,dz \quad , n=3$

$= \frac{2\pi i}{3!} f'''(z_0)$ where,

$$\boxed{\begin{aligned} f(z) &= \sin 2z \\ f'(z) &= 2\cos 2z \\ f''(z) &= -4\sin 2z \\ f'''(z) &= -8\cos 2z \\ f'''(z_0) = f'''(0) &= -8\cos 0 \\ &= -8. \end{aligned}}$$

$= \frac{2\pi i}{3!} \times -8$

$= \frac{2\pi i}{6} \times -8$

$= \frac{-8}{3}\pi i$

Q) Calculate $\oint_C \frac{z^2+5z+3}{(z-2)^2}$; C is $|z|=3$

Soln:

$z=2$ is a singular point.

when $z=2$, $|z| = 2 < 3$

$\therefore z_0 = 2$ lies inside C.

By CIF, $\oint_C \frac{z^2+5z+3}{(z-2)^2}\,dz = \oint_C \frac{z^2+5z+3}{(z-2)^{1+1}}\,dz$

$$\boxed{\text{CIF} \Rightarrow \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} \times f^{n}(z_0).}$$

$\therefore \oint_C \frac{z^2+5z+3}{(z-2)^{1+1}} = \frac{2\pi i}{1!} \times f'(z_0)$, where $f(z) = z^2+5z+3$

$f'(z) = 2z+5$

$= 2\pi i \times 9$

$f'(z_0) = \quad =$

$= 18\pi i$

$f(z_0) \Rightarrow 9$

a) Evaluate $\oint_c \frac{e^z}{z(1-z)^3}\,dz$; c is $|z| = \frac{1}{2}$

Soln:

$z = 0, 1$ are singularities. ; c ; $|z| = \frac{1}{2}$

when $z=0$, $|z| = 0 < \frac{1}{2} \Rightarrow$

when $z=1$, $|z| = 1 > \frac{1}{2}$

$\therefore$ $z=0$ lies inside c and $z=1$ lies outside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i \times f(z_0)$

$$\therefore \oint_c \frac{e^z}{z(1-z)^3}\,dz = \oint_c \frac{e^z/(1-z)^3}{z}\,dz = 2\pi i \times f(0)$$

$$\boxed{\begin{aligned} &\text{where } f(z) = \frac{e^z}{(1-z)^3} \\ &f(z_0) = f(0) = \frac{e^0}{1-0} = \underline{\underline{1}} \end{aligned}}$$

$$\therefore \oint_c \frac{e^z}{z(1-z)^3}\,dz = 2\pi i \times 1 = \underline{\underline{2\pi i}}$$

---

2) Evaluate $\oint_c \frac{z^2+2z+3}{z^2-1}\,dz$; c is $|z-1| = 1$

Soln:

Here $z = \pm 1$ are singular points.

$(z^2-1 = (z+1)(z-1))$

when $z=+1$, $|z-1| = |1-1| = 0 < 1$

when $z=-1$, $|z-1| = |-1-1| = 2 > 0$

$\therefore$ $z=+1$ lies inside C and $z=-1$ lies outside C.

$\therefore$ By CIF, $\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i \times f(z_0)$

$$\Rightarrow \oint_C \frac{z^2+2z+3}{(z^2-1)}\,dz = \oint_C \frac{z^2+2z+3}{(z+1)(z-1)} = \oint_C \frac{\left(z^2+2z+3/_{z+1}\right)}{z-1}$$

$$\Rightarrow \oint_C \frac{\left(\frac{z^2+2z+3}{z+1}\right)}{z-1}\,dz = 2\pi i \times f(z_0)$$

$$= 2\pi i \times f(1)$$

$$= 2\pi i \times 3$$

$$= \underline{\underline{6\pi i}}$$

$$\boxed{\begin{aligned} f(z) &= \left(\frac{z^2+2z+3}{z+1}\right) \\ \therefore f(1) &= \frac{6}{2} = 3 \end{aligned}}$$

Q) Using CIF, evaluate $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$; $C; |z|=1$

Soln:

$$z^4+2iz^3 = 0 \quad \Rightarrow z^3(z+2i) = 0$$

$$\Rightarrow z = 0 \text{ and } z = -2i$$

$z=0, -2i$ are singularities.

when $z=0$, $|z| = |0| = 0 < 1$

when $z=-2i$, $|z| = |-2i| = \sqrt{4} = 2 > 1$

$\therefore$ $z=0$ lies inside C and $z=-2i$ lies outside C.

$\therefore$ By CIF, $\oint_C \frac{z+1}{z^4+2iz^3}\,dz = \oint_C \frac{z+1}{z^3(z+2i)}\,dz$

$$= \oint_C \frac{\left(z+1/_{z+2i}\right)}{(z-0)^3}\,dz = \oint_C \frac{\left(\frac{z+1}{z+2i}\right)}{(z-0)^{2+1}}\,dz$$

$$= \frac{2\pi i}{2!} \times f''(z_0)$$

where $f(z) = \frac{z+1}{z+2i}$

$$f'(z) = \frac{(z+2i)\cdot 1 - (z+1)\cdot 1}{(z+2i)^2} = \frac{2i-1}{(z+2i)^2}$$

$$f''(z) = \frac{d}{dz}\frac{(2i-1)}{(z+2i)^2}.$$

$$= (2i-1) \times \frac{-1}{(z+2i)^4} \times 2(z+2i)$$

$$\Rightarrow f''(z) = (2i-1)\cdot \frac{-2}{(z+2i)^3}$$

$$\Rightarrow f(0) = (2i-1)\cdot\frac{-2}{(2i)^3} = (2i-1)\cdot -2 \cdot \frac{1}{8i^{-i}}$$

$$\Rightarrow \quad f(0) = \frac{(2i-1)}{4i}$$

$$\therefore \oint_c \frac{\left(z+1/z+2i\right)}{(z-0)^3}\,dz = \frac{2\pi i}{2!} \times f(0)$$

$$= \frac{2\pi i}{2!} \times \frac{(2i-1)}{4i}$$

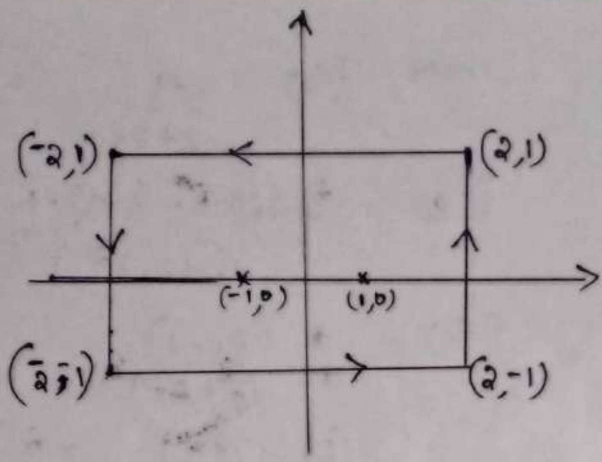
$$= \frac{\pi}{4}(2i-1)$$

Q) Evaluate $\oint_c \frac{\cos \pi z}{z^2-1}\,dz$; c is the rectangle with vertices $2 \pm i$, $-2 \pm i$.

Soln:

$$z^2 - 1 = 0 \Rightarrow (z+1)(z-1) = 0 \Rightarrow z = -1, z = 1$$

$\therefore z = \pm 1$ are singularities.



Here both the points $z = +1$ and $z = -1$ lies inside c.

$$\therefore \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

Using partial fractions,

$$\frac{1}{(z+1)(z-1)} = \frac{A}{(z+1)} + \frac{B}{(z-1)}$$

$$\Rightarrow 1 = A(z-1) + B(z+1)$$

put $z = 1 \Rightarrow 1 = 2B$

$$\Rightarrow \boxed{B = \frac{1}{2}}$$

put $z = -1 \Rightarrow 1 = -2A$

$$\Rightarrow \boxed{A = \frac{-1}{2}}$$

$$\therefore \frac{1}{z^2-1} = \frac{-1/2}{z+1} + \frac{1/2}{z-1}$$

$$\therefore \frac{\cos\pi z}{z^2-1} = -1/2 \frac{\cos\pi z}{z+1} + 1/2 \cdot \frac{\cos\pi z}{z-1}$$

$$\oint_c \frac{\cos\pi z}{z^2-1}\,dz = \frac{-1}{2}\oint_c \frac{\cos\pi z}{z+1}\,dz + \frac{1}{2}\oint_c \frac{\cos\pi z}{z-1}\,dz$$

$$= \left[\frac{-1}{2} \times 2\pi i \times f(-1)\right] + \left[\frac{1}{2} \times 2\pi i \times f(1)\right]$$

when $f(z) = \cos \pi z$

$f(-1) = \cos \pi = (-1) = -1$

$f(1) = \cos \pi = -1$

$$\therefore \oint_c \frac{\cos \pi z}{z^2-1} dz = \frac{-1}{2} \times 2\pi i \times (-1) + \frac{1}{2} \times 2\pi i \times (-1)$$

$$= \pi i - \pi i$$

$$= \underline{\underline{0}}$$

---

2) Using CIF, Calculate $\oint_c \frac{4-3z}{z^3-z^2-z+1} dz$

where C is (i) $|z+1| = \frac{3}{2}$

ii) $|z-1-i| = \frac{\pi}{2}$

<u>Soln:</u>

$z^3-z^2-z+1=0 \Rightarrow z^2(z-1)-(z-1)=0$

$\Rightarrow (z-1)[z^2-1] = 0$

$\Rightarrow (z-1)(z+1)(z-1) = 0$

$\therefore$ $z = \pm 1$ are singular points.

i) $|z+1| = \frac{3}{2}$

when $z=1$, $|z+1| = |2| = 2 > \frac{3}{2}$

when $z=-1$, $|z+1| = |-1+1| = 0 < \frac{3}{2}$

$\therefore$ $z=1$ lies outside C and $z=-1$ lies inside C.

$$\therefore \oint_c \frac{z^2}{z^3-z^2-z+1} dz = \oint_c \frac{z^2}{(z-1)^2(z+2)} dz$$

$$\Rightarrow \oint_C \frac{\left(z^2/(z-1)^2\right)}{z+1} \cdot dz = 2\pi i \times f(z_0)$$

where $f(z) = \dfrac{z^2}{(z-1)^2}$

$f(z_0) = f(-1) = \dfrac{1}{4}$

$$\therefore \oint_C \frac{z^2/(z-1)^2}{z+1} \, dz = 2\pi i \times \frac{1}{4}$$

$$= \frac{\pi i}{2}$$

ii) $C;\ |z-1-i| = \dfrac{\pi}{2}$

when $z=1$, $|z-1-i| = |1-1-i| = |-i| = \sqrt{1} = 1 < \dfrac{\pi}{2}$

when $z=-1$, $|z-1-i| = |-1-1-i| = |-2-i| = \sqrt{-2^2+-1^2} = \sqrt{5} > \dfrac{\pi}{2}$

$\therefore$ $z=1$ lies inside c and $z=-1$ lies outside c.

$$\therefore \oint_C \frac{z^2}{z^3-z^2-z+1} dz = \oint_C \frac{z^2}{(z-1)^2(z+1)} = \oint_C \frac{\left(z^2/z+1\right)}{(z-1)^2} dz.$$

By CIF $\Rightarrow$ $\displaystyle\oint \frac{\left(z^2/z+1\right)}{(z-1)^2} dz = \frac{2\pi i}{1!} \times f'(z_0)$

$= 2\pi i \times f'(1)$

$= 2\pi i \times \dfrac{3}{4}$

$= \dfrac{3\pi i}{2}$

where $f(z) = \dfrac{z^2}{z+1}$

$f'(z) = \dfrac{(z+1)2z - z^2(1)}{(z+1)^2}$

$f'(z_0) = f'(1) = \dfrac{3}{4}$

TAYLOR AND MACLAURIN SERIES

TAYLOR'S SERIES

If $f(z)$ is analytic in a closed curve c; $|z-a| = r$, then Taylor's series expansion of $f(z)$ at $z=a$ is
$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \cdots$$

Maclaurin Series

at $z=0$, then Maclaurin series expansion of $f(z)$ is,
$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \cdots$$

※ Important Results

① $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$

② $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots$

③ $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

④ $\log(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \cdots$

⑤ $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$ if $|z|<1$

⑥ $(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots$ if $|z|<1$

⑦ $(1+z)^{-2} = 1-2z+3z^2-4z^3+ \cdots$ if $|z|<1$

⑧ $(1-z)^{-2} = 1+2z+3z^2+4z^3+ \cdots$ if $|z|<1$

Q) Find the Taylor series expansion of $f(z) = \frac{1}{z^2}$ about $z=2$.

Soln:

Taylor's series expansion of $f(z)$ is,

$$f(z) = f(a) + f'(a)$$

Here $f(z) = \frac{1}{z^2}$

at $z=2$, $f(z) = \frac{1}{z^2} = \frac{1}{(2+z-2)^2} = \frac{1}{2^2\left(1+\frac{z-2}{2}\right)^2}$

$$\Rightarrow f(z) = \frac{1}{4\left(1+\frac{z-2}{2}\right)^2}$$

$$= \frac{1}{4}\left[1+\frac{z-2}{2}\right]^{-2}$$

$$= \frac{1}{4}\left[1+{}^{-}2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2-4\left(\frac{z-2}{2}\right)^3+\cdots\right]$$

, if $|z|<1$

$$= \frac{1}{4}\left[1-(z-2)+3\frac{(z-2)^2}{4}-4\frac{(z-2)^3}{8}+\cdots\right]$$

if $|\cancel{z}|<1$

$$\boxed{\text{Since } (1+z)^{-2} = 1-2z+3z^2-4z^3+\cdots \text{ if } |z|<1}$$

Q) Find the Taylor series of $\frac{\sin z}{z-\pi}$ about the point $z=\pi$.

Soln:

put $z-\pi = t \Rightarrow z = \pi + t$

$\therefore f(z) = \frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{z-\pi} = \frac{-\sin t}{t}$ $\quad \because \sin(\pi+\theta) = -\sin\theta$

$\therefore \Rightarrow \frac{-\sin t}{t} = \frac{-1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]$

$$\boxed{\text{Using the result, } \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$\Rightarrow f(z) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \frac{t^6}{7!} - \cdots$

$\Rightarrow f(z) = -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \frac{(z-\pi)^6}{7!} - \cdots,$

$|z-\pi| < 1$

Q) Find the Taylor series of $f(z) = \frac{z+1}{z-1}$ about $z=-1$.

Soln:

$f(z) = \frac{z+1}{z-1}$

$= \frac{z+1}{z-1+1-1} = \frac{z+1}{z+1-2} = \frac{z+1}{-2\left(1-\frac{z+1}{2}\right)}$

$= \frac{z+1}{-2}\left(1 - \frac{(z+1)}{2}\right)^{-1}$

$= \frac{z+1}{-2}\left[1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \left(\frac{z+1}{2}\right)^3 + \cdots\right]$

$$= -\left[\frac{z+1}{2} + \frac{(z+1)^2}{2} + \frac{(z+1)^3}{3} + \frac{(z+1)^4}{4} + \cdots \right]$$

(e) Write the Maclaurin series expansion of the $f(z) = \frac{1}{z+1}$ at $z=0$

Soln:

Maclaurin series expansion of $f(z)$ is

$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \cdots \quad \text{— (1)}$$

We've, $f(z) = \frac{1}{z+1}$, $f(0) = \frac{1}{0+1} = 1$

$$f'(z) = \frac{-1}{(z+1)^2}, \quad f'(0) = \frac{-1}{1} = -1$$

$$f''(z) = \frac{-1 \times -2}{(z+1)^3}, \quad f''(0) = \frac{2}{1^3} = 2$$

$$f'''(z) = \frac{-6}{(z+1)^4}, \quad f''(0) = \frac{-6}{1^4} = -6$$

Substituting all these values in $f(z)$ (1).

$$\therefore (1) \Rightarrow f(z) = 1 + -1.z + 2.\frac{z^2}{2!} + -6 \times \frac{z^3}{3!} + \cdots$$

$$\therefore \frac{1}{z+1} = f(z) = 1 - z + z^2 - z^3 + \cdots$$

MODULE V

RESIDUE INTEGRATION

LAURENT'S SERIES

If $f(z)$ is analytic within or on annulus, then for all z in that region,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots +$$
$$\cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

Note:

① $(1+z)^{-1} = 1 - z + z^2 - \cdots\cdots = \sum_{n=0}^{\infty} (-1)^n z^n$

② $(1-z)^{-1} = 1 + z + z^2 + \cdots\cdots = \sum_{n=0}^{\infty} z^n$

③ $(1+z)^{-2} = 1 - 2z + 3z^2 - \cdots\cdots = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n$

④ $(1-z)^{-2} = 1 + 2z + 3z^2 + \cdots\cdots = \sum_{n=0}^{\infty} (n+1) z^n$

These expansions are valid if $|z| < 1$.

Q) Expand $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in Laurents series for the region i) $|z| < 2$ ii) $2 < |z| < 3$

Soln:

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

Now $\quad f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{-5z-7}{(z+2)(z+3)}$

Now $\quad \dfrac{-5z-7}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3}$

$\Rightarrow -5z-7 = A(z+3) + B(z+2)$

put $z = -3 \Rightarrow -B = 8 \Rightarrow \boxed{B = -8}$

put $z = -2 \Rightarrow A = 3 \Rightarrow \boxed{A = 3}$

$\therefore \dfrac{-5z-7}{(z+2)(z+3)} = \dfrac{3}{z+2} + \dfrac{-8}{z+3}$

$\therefore f(z) = 1 + \dfrac{-5z-7}{(z+2)(z+3)}$

ie $\quad f(z) = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

i) $|z| < 2 \Rightarrow \left|\dfrac{z}{2}\right| < 1$

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$\Rightarrow \quad f(z) = 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n;$$

This expansion is valid in $\left|\dfrac{z}{2}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$

$\Rightarrow |z| < 2$ and $|z| < 3$

$\Rightarrow \underline{|z| < 2}$

ii) $2 < |z| < 3 \quad \Rightarrow |z| > 2$ and $|z| < 3$

$\Rightarrow \frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$|z| > 2$

$\Rightarrow \frac{|z|}{2} > 1$

$\Rightarrow \frac{2}{|z|} < 1$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n ;$$

This expansion valid only if, $\left|\frac{2}{z}\right| < 1$ and $\frac{|z|}{3} < 1$

ie $2 < |z|$ and $|z| < 3$

ie $2 < |z| < 3$

Q) Find the Laurent's expansion of $\frac{1}{z-z^3}$ in $1 < |z+1| < 2$

Soln:

$$f(z) = \frac{1}{z-z^3} = \frac{1}{z(1-z^2)} = \frac{1}{z(1+z)(1-z)}$$

$$\frac{1}{z(1+z)(1-z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{1-z}$$

$$\Rightarrow 1 = A(1+z)(1-z) + Bz(1-z) + C(z)(1+z)$$

Put $z=0 \Rightarrow \boxed{A = 1}$

put $z=1 \Rightarrow 2C = 1 \Rightarrow \boxed{C = \frac{1}{2}}$

put $z=-1 \Rightarrow 1 = -2B \Rightarrow \boxed{B = \frac{-1}{2}}$

$$\therefore f(z) = \frac{1}{z} + \frac{-1/2}{1+z} + \frac{1/2}{1-z} \quad ; \quad 1<|z+1|<2$$

$$\therefore f(z) = \frac{1}{(z+1-1)} + \frac{-1/2}{(1+z)} + \frac{1/2}{-(z+1-2)}$$

$$= \frac{1}{(u-1)} + \frac{-1/2}{(u)} + \frac{-1/2}{(u-2)} \quad ; \quad u = z+1$$

$$= \frac{1}{u-1} + \frac{-1}{2}\cdot\frac{1}{u} - \frac{1}{2}\cdot\frac{1}{u-2}$$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{-1}{2}\cdot\frac{1}{u} - \frac{1}{2}\cdot\frac{1}{-2\left(1-\frac{u}{2}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{2u} + \frac{1}{4}\left(1-\frac{u}{2}\right)^{-1}$$

$$\therefore f(z) = \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n - \frac{1}{2u} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n ;$$

valid in $\left|\frac{1}{u}\right|<1$ and $\left|\frac{u}{2}\right|<1$.

$$\Rightarrow 1<|u| \text{ and } |u|<2$$

$$\Rightarrow 1<|u|<2$$

$$\Rightarrow 1<|z+1|<2$$

$$\therefore f(z) = \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{1}{2(z+1)} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n,$$

if $1<|z+1|<2$

Q) Find the Laurent's expansion of $\frac{1}{z(1-z)}$ valid in the region $|z+1| > 2$.

Soln:

$$f(z) = \frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$

$$\Rightarrow 1 = A(1-z) + Bz$$

When $z=0$, $1 = A \Rightarrow \boxed{A=1}$

When $z=1$, $1 = B \Rightarrow \boxed{B=1}$

$$\therefore f(z) = \frac{A}{z} + \frac{B}{1-z}$$

ie $$f(z) = \frac{1}{z} + \frac{1}{1-z}$$

$$= \frac{1}{z+1-1} + \frac{1}{-(z-1)}$$

$$= \frac{1}{z+1-1} + - \frac{1}{z+1-2}$$

$$= \frac{1}{u-1} - \frac{1}{u+2} \quad ; \text{Put } z+1 = u$$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{1}{u\left(1+\frac{2}{u}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1+\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^{n} - \frac{1}{u}\sum_{n=1}^{\infty}\left(\frac{2}{u}\right)^{n}$$

valid in $\left|\frac{1}{u}\right| < 1$ and $\left|\frac{2}{u}\right| < 1$

Q) Find the Laurent's series of $z^{-5} \sin z$ with centre '0'.

<u>Soln</u>:

given, $f(z) = z^{-5} \sin z$

$\because \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$$= z^{-5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^4} - \frac{1}{z^2}\cdot\frac{1}{3!} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots$$

with centre '0'.

Q) Expand $f(z) = \dfrac{z}{(z+1)(z+2)}$ in Laurent's series about $z = -2$.

Soln:

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

put $z = -1 \Rightarrow -1 = A$

$$\Rightarrow \boxed{A = -1}$$

put $z = -2 \Rightarrow -2 = -B$

$$\Rightarrow \boxed{B = 2}$$

$$\therefore f(z) = \frac{A}{z+1} + \frac{B}{z+2}$$

$$= \frac{-1}{z+1} + \frac{2}{z+2}$$

given Region is about $z = -2$

ie $z+2$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2} \text{ becomes}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= -\frac{1}{-1[1-(z+2)]} + \frac{2}{z+2}.$$

$$= \frac{1}{[1-(z+2)]} + \frac{2}{(z+2)}.$$

$$= [1-(z+2)]^{-1} + \frac{2}{(z+2)}.$$

$$= \left[1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots\right] + \frac{2}{z+2}$$

$$= \left[\sum_{n=0}^{\infty} (z+2)^n\right] + \frac{2}{z+2}.$$

$$f(z) = \frac{2}{z+2} + \sum_{n=0}^{\infty} (z+2)^n \quad \text{about} \quad z = -2.$$

Q) Find the Laurent's series of $z^2 e^{1/z}$ with centre '0'.

Soln:

We've $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$\therefore$ for $e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!}$

But our given function is, $f(z) = z^2 e^{1/z}$

$$\therefore f(z) = z^2 e^{1/z}$$

$$= z^2 \left[1 + \frac{(\frac{1}{z})}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots\right]$$

$$= z^2 + \frac{z^1}{1!} + \frac{1}{2!} + \frac{1}{z} \cdot \frac{1}{3!} + \cdots$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6} z^{-1} + \cdots$$

$$\Rightarrow f(z) = (z-0)^2 + (z-0) + \frac{1}{2} + \frac{1}{6}(z-0)^{-1} + \cdots$$

with centre '0'.

Q) Find the Laurents series expansion of

$$f(z) = \frac{z}{(z^2-1)(z^2+4)}, \quad 1<|z|<2.$$

Soln:

$$f(z) = \frac{z}{(z^2-1)(z^2+4)} = \frac{z}{(z+1)(z-1)(z+2i)(z-2i)}$$

$$\Rightarrow \frac{z}{(z+1)(z-1)(z+2i)(z-2i)} = \frac{A}{z+1} + \frac{B}{(z-1)} + \frac{C}{(z+2i)} + \frac{D}{(z-2i)}$$

$$\Rightarrow z = A(z-1)(z+2i)(z-2i) + B(z+1)(z+2i)(z-2i) + C(z+1)(z-1)(z-2i) + D(z+1)(z-1)(z+2i)$$

put $z=1 \Rightarrow 1 = B(1+1)(1+2i)(1-2i)$

$\Rightarrow 1 = 2B(1+4)$

$\Rightarrow \boxed{B = 1/10}$

Put $z=-1 \Rightarrow -1 = A(-1-1)(-1+2i)(-1-2i)$

$\Rightarrow -1 = -2A(1+4)$

$\Rightarrow \boxed{A = 1/10.}$

Put $z=2i \Rightarrow D(2i+1)(2i-1)(2i+2i) = 2i$

$\Rightarrow D(-4-1)4i = 2i$

$\Rightarrow \boxed{D = -1/10}$

Put $z=-2i \Rightarrow -2i = C(-2i+1)(-2i-1)(-2i-2i)$

$\Rightarrow -2i = C(-4-1)-4i$

$\Rightarrow \boxed{C = -1/10}$

$$\therefore f(z) = \frac{1/10}{z+1} + \frac{1/10}{z-1} + \frac{-1/10}{z+2i} + \frac{-1/10}{z-2i}$$

$= \frac{\cancel{1}}{\cancel{10}}\ \cancel{\pm}$ given region is $1<|z|<2$

$\Rightarrow 1<|z|$ and $|z|<2$

$\Rightarrow \frac{1}{|z|}<1$ and $\left|\frac{z}{2}\right|<1$

$$\therefore f(z) = \frac{1}{10}\cdot\frac{1}{z(1+\frac{1}{z})} + \frac{1}{10}\cdot\frac{1}{z(1-\frac{1}{z})} + \frac{-1}{10}\cdot\frac{1}{2i(1+\frac{z}{2i})}$$

$$+ \frac{-1}{10}\cdot\frac{1}{2i(1-\frac{z}{2i})}$$

$$= \frac{1}{10}\,\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} + \frac{1}{10}\,\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{20i}\left(1+\frac{z}{2i}\right)^{-1} + \frac{1}{20i}\left(1-\frac{z}{2i}\right)^{-1}$$

$$= \frac{1}{10}\cdot\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n + \frac{1}{10}\cdot\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{20i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n$$

$$+\frac{1}{20i}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n$$

valid in $\frac{1}{|z|}<1$ and $\left|\frac{z}{2i}\right|<1$

ie $\frac{1}{|z|}<1$ and $\left|\frac{z}{2}\right|<1$

ie $1<|z|<2$.

CLASSIFICATION OF SINGULARITIES

If $f(z)$ is not analytic at $z=a$, then $z=a$ is called singular point or singularity of $f(z)$.

Eg: $f(z) = \frac{1}{z+2}$; $z=-2$ is singular point.

$f(z) = \frac{1}{z(z+1)(z+2)}$; $z=0,-1,2$ are singular points.

① ISOLATED SINGULAR POINT:-

A singular point $z=a$ of a function $f(z)$ is called an Isolated singular point if there exist a circle with centre at 'a' which contains no other singular points of $f(z)$.

Eg: 1) $f(z) = \frac{z}{z^2-1}$; $z=\pm 1$ are ~~sing~~ Isolated singular points.

2) $f(z) = \frac{1}{\sin \pi z}$; $\sin \pi z = 0 \Rightarrow \pi z = \pm n\pi$

$\Rightarrow z = \pm n$ $\quad \because \sin n\pi = 0$

$\therefore f(z)$ has infinite no. of isolated singular points.

3) $f(z) = \tan z$; $\Rightarrow f(z) = \frac{\sin z}{\cos z}$; $\cos z = 0$

$\Rightarrow z = \pm(2n+1)\frac{\pi}{2}$

$\because \cos(2n+1)\frac{\pi}{2} = 0.$

$$\Rightarrow \quad z = \pm(2n+1)\frac{\pi}{2}$$

$\therefore$ f(z) has infinitely many isolated singularity.

(4) NON- ISOLATED SINGULAR POINT

Eg: $f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin(1/z)}{\cos(1/z)}$; $\cos\left(\frac{1}{z}\right) = 0$

$$\Rightarrow \quad \frac{1}{z} = \pm(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad z = \pm\frac{2}{(2n+1)\pi}$$

Here $z = 0$ is a non-isolated singular point.

(2) REMOVABLE SINGULARITIES

Suppose $z=a$ is an isolated singular point of $f(z)$ then we can express $f(z)$ as a Laurent's series about $z=a$ as,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \text{------} + \text{--}$$
$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \text{-----}$$

The first series is called Taylor part and the second series is called principal part. If the principal part is absent, $z=a$ is called removable singular point.

~~Eg~~ (3) ESSENTIAL SINGULARITY

If there are infinite no. of terms in the principal part, $z=a$ is called an Essential singularity.

(4) POLE

If there are finite no. of terms in the principal part, $z=a$ is called pole.

The highest power of $\frac{1}{z-a}$ is called ~~pole~~ order.

pole of order 1 is called simple pole.

Q) Determine and classify the singular points of the following functions.

1) $f(z) = \frac{1}{(z-3)(z-1)}$

$z=3$ and $z=1$ are isolated singularities.

Both are simple poles.

2) $f(z) = \cot\left(\frac{\pi}{z}\right)$

$= \frac{\cos(\pi/z)}{\sin(\pi/z)}$

$\sin\frac{\pi}{z} = 0$

$\Rightarrow \frac{\pi}{z} = \pm n\pi$

$$\Rightarrow z = \pm \frac{1}{n}$$

Here $z=0$ is a non-isolated singularity.

and $z = \pm \frac{1}{n}$ are isolated singularities.

---

3) $f(z) = e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$

$$= 1 + \frac{1}{z} + \frac{1}{z^2}\cdot\frac{1}{2!} + \frac{1}{z^3}\cdot\frac{1}{3!} + \cdots$$

Here $z=0$ is a singular point. Since the principal part contains infinite no. of terms.

$\therefore$ $z=0$ is Essential singularity.

---

4) $f(z) = \frac{1}{(z-3)^3(z+6)}$

Here $z=3$ and $z=-6$ are singularities.

$z=3$ is a pole of order 3. and

$z=-6$ is a pole of order 1. ie simple pole.

---

5) $f(z) = \frac{z - \sin z}{z^3}$

$z=0$ is a singular point.

ie $f(z) = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right]$

$$\therefore f(z) = \frac{1}{z^3}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} + \cdots$$

Since the principal part is absent, $z=0$ is a removable singularity.

---

6) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

$\Rightarrow \cos z = 0 \Rightarrow z = \pm(2n+1)\dfrac{\pi}{2}$

all are isolated singular points.

---

7) $f(z) = e^{-1/z^2}$

$$f(z) = e^{-1/z^2} = 1 - \frac{(1/z^2)}{1!} + \frac{(1/z^2)^2}{2!} \mp \cdots$$

$$= 1 - \frac{1}{z^2} + \frac{1}{2!\,z^4} + \cdots$$

Here $z=0$ is singularity.

Since principal part contains infinite no. of terms, $z=0$ is Essential singularity.

---

8) $f(z) = \dfrac{\sin z}{(z-\pi)^2}$

$\Rightarrow f(z) = \dfrac{\sin z}{(z-\pi)^2}$

$\left[\because \sin(\pi+\theta) = -\sin\theta\right.$

$\Rightarrow z=\pi$ is a singular point.

$$\therefore f(z) = \frac{\sin z}{(z-\pi)^2} = \frac{\sin(z-\pi+\pi)}{(z-\pi)^2}$$

$$= \frac{\sin[\pi+(z-\pi)]}{(z-\pi)^2} = \frac{-\sin(z-\pi)}{(z-\pi)^2}$$

$$= \frac{-1}{(z-\pi)^2}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots\cdots\right]$$

$\left[\because \sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots\right.$

$$f(z) = -\frac{1}{(z-\pi)} + \frac{(z-\pi)}{3!} - \frac{(z-\pi)^3}{5!} + \cdots\cdots$$

Here principal part contains only one term ie $\left(\frac{1}{z-\pi}\right)$ term. ie finite no. of term in principal part.

$\therefore$ $z=\pi$ is a pole of order '1'.

ie $z=\pi$ is a simple pole.

(9) $f(z) = e^{\frac{1}{z-1}}$

$$\therefore f(z) = e^{\frac{1}{z-1}} = 1 + \frac{\left(\frac{1}{z-1}\right)}{1!} + \frac{\left(1/z-1\right)^2}{2!} + \frac{\left(1/z-1\right)^3}{3!} + \cdots$$

$$= 1 + \frac{1}{(z-1)} + \frac{1}{2!}\frac{1}{(z-1)^2} + \frac{1}{3!}\frac{1}{(z-1)^3} + \cdots$$

$\Rightarrow z = 1$ is a singular point.

Since principal part contains infinite no. of terms, $z=1$ is an Essential singularity.

ie $z=1$ is Essential singular point.

---

(10) $f(z) = \dfrac{1-\cos z}{z^2}$.

$z = 0$ is a singular point.

$$f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$$

$$\because \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$= \frac{1}{z^2}\left[\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right]$$

$$= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots$$

Here principal part is absent.

$\therefore$ $z=0$ is Removable singularity.

(11) $f(z) = \dfrac{\sin z}{z^4}$

Here $z=0$ is a singular point.

$$\therefore f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^3} - \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{5!}z - \frac{1}{7!}z^3 + \cdots$$

Here principal part contains finite no of terms. ie only one term $\frac{1}{z^3}$

$\therefore$ $z=0$ is a pole. of order 3.

RESIDUE OF A FUNCTION

Residue of a function $f(z)$ at $z=a$ is the Coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ and is denoted by $\underset{z=a}{\text{Res}}\, f(z)$.

CALCULATION OF RESIDUE.

① If $f(z)$ has a simple pole at $z=a$,

then $$\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a} (z-a)\, f(z)$$

or

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)}$$, where $f(z) = \frac{g(z)}{h(z)}$ and $g(a) \neq 0$.

② If $f(z)$ has a pole of order m at $z=a$,

then $$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

, $m \geq 2$

Q) Find the singularities and Residue of $f(z) = \frac{1}{z^4 - 1}$

Soln:

$$f(z) = \frac{1}{z^4-1} = \frac{1}{(z^2+1)(z^2-1)} = \frac{1}{(z+1)(z-1)(z+i)(z-i)}$$

The singularities are $z = 1, -1, i, -i$.

all are simple poles.

at $z=1$,

$$\underset{z=1}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(1)}{h'(1)} \quad \text{where } g(z) = 1$$

$$g(a) = g(1) = 1$$

$$= \frac{1}{4} \qquad \text{also } h(z) = z^4 - 1$$

$$h'(z) = 4z^3$$

$$h'(a) = h'(1) = 4.$$

at $z=-1$,

$$\underset{z=-1}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(-1)}{h'(-1)}$$

$$g(a) = g(-1) = 1$$

$$= \frac{1}{-4} \qquad h'(a) = h'(-1) = -4$$

at $z=i$,

$$\underset{z=i}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(i)}{h'(i)}$$

$$g(a) = g(i) = 1$$

$$= \frac{1}{-4i} \qquad h'(a) = h'(i) = 4i^3 = -4i$$

$$= \frac{i}{4} \qquad \left\langle \text{since } \frac{1}{i} = -i \right.$$

at $z=-i$

$$\operatorname*{Res}_{z=-i} f(z) = \frac{g(a)}{h'(a)} = \frac{g(-i)}{h'(-i)}$$

$g(a) = g(-i) = 1$

$$= \frac{1}{4i}$$

$h'(a) = h'(-i) = 4(-i)^3$

$= 4 \times -1 \times i^3$

$= 4i$

$$= \frac{-i}{4} \qquad \left(\because \frac{1}{i} = -i\right)$$

Q) Find the residue of $f(z) = \dfrac{ze^z}{z^2+4}$

Soln:

$z^2+4=0 \Rightarrow (z+2i)(z-2i)=0$

$\therefore$ Singular points are $z=+2i$ and $z=-2i$

both are simple poles.

$$\therefore f(z) = \frac{ze^z}{z^2+4} = \frac{ze^z}{(z+2i)(z-2i)}$$

To find residue,

Residue at $z=2i$,

$$\operatorname*{Res}_{z=2i} f(z) = \frac{g(a)}{h'(a)}$$

where $g(z) = ze^z$

$g(a) = g(2i) = 2ie^{2i}$

$h(z) = z^2+4$

$h'(z) = 2z$

$h'(a) = h'(2i) = 4i$

$$= \frac{2ie^{2i}}{4i}$$

$$= \frac{e^{2i}}{2}$$

Residue at $z=-2i$,

$$\operatorname*{Res}_{z=-2i} f(z) = \frac{g(a)}{h'(a)}$$

$$= \frac{-2i e^{-2i}}{-4i}$$

$$= \frac{e^{-2i}}{2}$$

where $g(z) = z e^{z}$

$g(a) = g(-2i) = -2i e^{-2i}$

$h(z) = z^2 + 4$

$h'(z) = 2z$

$h'(a) = h'(-2i) = -4i$

---

2) Find the residue and singular points of $f(z) = \tan z$.

Soln:

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

To find singular points, $\cos z = 0$

$$\Rightarrow z = \pm(2n+1)\frac{\pi}{2}$$

when $n = 0, 1, \ldots$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

$\Rightarrow$ simple poles.

All are simple poles.

Residue at $z = \pm(2n+1)\frac{\pi}{2}$ is,

$$\operatorname*{Res}_{z=\pm(2n+1)\frac{\pi}{2}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=\pm(2n+1)\frac{\pi}{2}}$$

$$= \left.\frac{\sin z}{-\sin z}\right|_{z=\pm(2n+1)\frac{\pi}{2}}$$

$$= -1$$

where $f(z) = \frac{g(z)}{h(z)}$

$g(z) = \sin z$

$h(z) = \cos z$

2) Find the residue of $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

Soln:

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} = \frac{z^2 - 2z}{(z+1)^2 (z+2i)^1 (z-2i)^1}$$

Here singularities are,

$z = -1$ is a pole of order 2.

$z = -2i$ is a pole of order 1. ie simple pole.

$z = 2i$ is a pole of order 1. ie simple pole.

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right], \quad m=2$$

$$= \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z+1)^2} \cdot \frac{z^2 - 2z}{\cancel{(z+1)^2} (z^2+4)} \right]$$

$$= \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} \left[\frac{z^2 - 2z}{z^2 + 4} \right]$$

$$= \lim_{z \to -1} \left[\frac{(z^2+4)(2z-2) - (z^2 - 2z)\, 2z}{(z^2+4)^2} \right]$$

$$= \left[\frac{((-1)^2+4)(2 \times -1 - 2) - ((-1)^2 - 2 \times -1) \cdot 2 \times -1}{[(-1)^2 + 4]^2} \right]$$

$$= \left[\frac{(5 \times -4) + 6}{25} \right] = \frac{-14}{25}$$

$$\therefore \operatorname*{Res}_{z=-1} f(z) = \frac{-14}{25} \;//.$$

$z = 2i$ is a simple pole.

Now $\underset{z=2i}{\operatorname{Res}} f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=2i}$

$= \dfrac{g(a)}{h'(a)} = \dfrac{g(2i)}{h'(2i)}$, where,

$g(z) = z^2 - 2z$

$\therefore g(a) = g(2i)$

$= (2i)^2 - 2\times 2i$

$\boxed{\therefore g(2i) = -4 - 4i}$

and $h(z) = (z+1)^2(z^2+4)$

$h'(z) = (z+1)^2 \times 2z + (z^2+4)\times 2(z+1)$

$h'(a) = h'(2i) =$

$= (2i+1)^2 \times 2\times 2i + ((2i)^2 + 4)\times$
$\times 2(2i+1)$

$= (2i+1)^2 . 4i + 0$

$\boxed{\therefore h'(2i) = 4i(2i+1)^2}$

$\therefore \underset{z=2i}{\operatorname{Res}} f(z) = \dfrac{(-4-4i)}{4i(2i+1)^2}$

$= \dfrac{-4-4i}{4i(-4+4i+1)}$

$= \dfrac{-4-4i}{4i(-3+4i)}$

$= \dfrac{-4-4i}{4i(4i-3)}$

Now Residue of $f(z)$ at $z = -2i$.

$z = -2i$ is a simple pole.

$\therefore \underset{z=-2i}{\operatorname{Res}} f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=-2i}$

$= \left.\dfrac{z^2 - 2z}{(z+1)^2 \times 2z + (z^2+4)\times 2(z+1)}\right|_{z=-2i}$

$= \dfrac{(-2i)^2 - 2\times(-2i)}{((-2i)+1)^2 \times 2\times(-2i) + ((-2i)^2+4)\times 2(-2i+1)}$

$$\therefore \operatorname*{Res}_{z=-2i} f(z) = \frac{-4+4i}{(-2i+1)^2 \times -4i + 0}$$

$$= \frac{-4+4i}{(-4+-4i+1) \times -4i}$$

$$\therefore \operatorname*{Res}_{z=-2i} f(z) = \frac{-4+4i}{(-4i-3).-4i}$$

RESH

Q) Calculate Residue of $f(z) = \dfrac{1}{z^2(1+z)}$

Soln:

singularities are $z=0$ and $z=-1$.

$z=0$ is a pole of order 2.

$z=-1$ is a pole of order 1.

Residue at $z=0$ (order 2, ie $m=2$)

$$\therefore \operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m \cdot f(z)\right]$$

$$\text{ie } \operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d^{2-1}}{dz^{2-1}} \left[\frac{(z-0)^2}{z^2(1+z)} \right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{1}{1+z}\right].$$

$$= \lim_{z\to 0}\left[\frac{-1}{(1+z)^2}\right] = \frac{-1}{(1+0)^2} = \underline{\underline{-1}}$$

ie $\underset{z=0}{\operatorname{Res}} f(z) = -1$

Now Residue at $z = -1$. (pole of order 1. simple pole.)

$\therefore \underset{z=a}{\operatorname{Res}} f(z) = \left. \dfrac{g(z)}{h'(z)} \right|_{z=a}$

ie $\underset{z=-1}{\operatorname{Res}} f(z) = \left. \dfrac{1}{z^2 \cdot 1 + (1+z)\, 2z} \right|_{z=-1}$

where,

$g(z) = 1$

$\therefore g(a) = g(-1) = 1$

~~$g(z)$ $h(z) = \dfrac{1}{z^2(1+z)}$~~

$h(z) = z^2(1+z)$

$h'(z) = z^2 \cdot 1 + (1+z) 2z$

$\therefore h'(-1) = -1^2 \cdot 1 + (1+-1) \times -2$

$= 1$

$= \dfrac{1}{(-1)^2 + (1+-1) \cdot 2 \cdot -1}$

$= \dfrac{1}{1}$

$= 1$

$\therefore \underset{z=-1}{\operatorname{Res}} f(z) = 1$

Q) Find the Residue of $f(z) = \dfrac{z^3 + 2z}{(z-i)^3}$

Soln:-

Singularities are $z = i$. (order 3. ie $m = 3$).

$\therefore \underset{z=a}{\operatorname{Res}} f(z) = \dfrac{1}{(m-1)!} \lim_{z \to a} \dfrac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right]$.

ie $\underset{z=i}{\text{Res}}\, f(z) = \dfrac{1}{(3-1)!} \lim\limits_{z \to i} \dfrac{d^{3-1}}{dz^{3-1}} \left[(z-i)^3 \cdot \dfrac{z^3+2z}{(z-i)^3} \right]$

$$= \frac{1}{2!} \lim_{z \to i} \frac{d^2}{dz^2} \left[z^3 + 2z \right]$$

$$= \frac{1}{2} \lim_{z \to i} \left[\frac{d}{dz} \left(3z^2 + 2 \right) \right]$$

$$= \frac{1}{2} \lim_{z \to i} \left[\cancel{2}\right] \left[6z \right]$$

$$= \frac{1}{2} \lim_{z \to i} \left[6z \right]$$

$$= \frac{1}{2} \left[6 \cdot i \right] = \underline{\underline{3i}}$$

$\therefore \underset{z=i}{\text{Res}}\, f(z) = \underset{z=i}{\text{Res}} \dfrac{z^3+2z}{(z-i)^3} = 3i$

CAUCHY'S RESIDUE THEOREM (CRT)

Evaluation of Integral

If $f(z)$ is analytic at all points inside and on a simple closed curve C except at a finite number of isolated singular points within C, then

$$\oint_C f(z)\, dz = 2\pi i \left[\text{Sum of residues at singular points within } C \right]$$

Q) Evaluate $\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$; c is $|z|=3$.

Using Cauchy's Residue theorem.

<u>Soln</u>:

Here $z=1$, and $z=2$ are singularities.

both are simple poles.

both lies inside C; $|z|=3$

$$\left[\begin{array}{l} \text{ie } |1| = 1 < 3 \text{ and} \\ \quad |2| = 2 < 3. \end{array}\right] \Rightarrow \text{lies inside } c.$$

$\therefore$ By Cauchy's Residue theorem,

$$\oint_C f(z)\, dz = 2\pi i\left[\text{sum of residues at points inside } c.\right]$$

~~Here Res~~ $= 2\pi i\left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z)\right]$

Now $\operatorname*{Res}_{z=1} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=1}$ $\qquad h(z) = (z-1)(z-2)$

$$= \left.\frac{\cos \pi z^2}{(z-1)\cdot 1 + (z-2)\cdot 1}\right|_{z=1}$$

$$= \frac{\cos \pi}{0 + -1\cdot 1} = \frac{\cos\pi}{-1} = \frac{-1}{-1}$$

ie $\operatorname*{Res}_{z=1} f(z) = \underline{\underline{1}}$.

Now $\operatorname*{Res}_{z=2} f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=2}$

$$= \left.\frac{\cos \pi z^2}{(z-1)\cdot 1 + (z-2)\cdot 1}\right|_{z=2}$$

$$= \frac{\cos \pi (4)}{1+0}$$

$$= \frac{\cos 4\pi}{1} = \frac{1}{1} \qquad \therefore \cos 4\pi = 1 \text{ n is even.}$$

ie $\operatorname*{Res}_{z=2} f(z) = \underline{\underline{1}}$

$\therefore$ By CRT,

$$\oint_c f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) \right]$$

$$= 2\pi i \left[+1 + 1 \right]$$

$$\therefore \oint_c \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz = \underline{\underline{4\pi i}}.$$

Q) Use Residue Theorem to evaluate $\oint_C \frac{30z^2-23z+5}{(2z-1)^2(3z-1)}\,dz$

where C ; $|z|=1$

Soln:

$(2z-1)=0 \Rightarrow z=\frac{1}{2}$

$(3z-1)=0 \Rightarrow z=\frac{1}{3}$

$z=\frac{1}{2}$ and $z=\frac{1}{3}$ are singularities.

$|z|=\left|\frac{1}{2}\right|=\frac{1}{2}<1$ lies inside C.

$|z|=\left|\frac{1}{3}\right|=\frac{1}{3}<1$ lies inside C.

both points lies inside C.

$\therefore$ By Cauchy's Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i \left[\text{sum of Residues at singularities inside } C.\right]$$

$$= 2\pi i\left[\operatorname*{Res}_{z=\frac{1}{2}} f(z) + \operatorname*{Res}_{z=\frac{1}{3}} f(z)\right]$$

Now Residue at $z=\frac{1}{2}$, [$z=\frac{1}{2}$ is a pole of order 2].

$$\operatorname*{Res}_{z=\frac{1}{2}} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right], \quad m=2.$$

$$= \frac{1}{(2-1)!}\lim_{z\to\frac{1}{2}}\frac{d^1}{dz}\left[\left(z-\frac{1}{2}\right)^2\cdot\frac{(30z^2-23z+5)}{(2z-1)^2(3z-1)}\right]$$

$$= \frac{1}{1!}\lim_{z\to\frac{1}{2}}\frac{d}{dz}\left[\cancel{\left(z-\frac{1}{2}\right)^2}\cdot\frac{(30z^2-23z+5)}{4\cancel{\left(z-\frac{1}{2}\right)^2}(3z-1)}\right]$$

$$= \frac{1}{4} \lim_{z \to \frac{1}{2}} \left[\frac{(3z-1)(60z-23) - (30z^2 - 23z + 5) \cdot 3}{(3z-1)^2} \right]$$

$$= \frac{1}{4} \left[\frac{\frac{1}{2} \cdot 7 - \left(\frac{15}{2} - \frac{23}{2} + 5 \right) \cdot 3}{1/4} \right]$$

$$= \frac{7}{2} - 3 = \frac{1}{2}$$

ie $\displaystyle \operatorname*{Res}_{z=\frac{1}{2}} f(z) = \frac{1}{2}$

Now Residue at $z = \frac{1}{3}$, $\left(\because z = \frac{1}{3} \right.$ is a simple pole

$$\operatorname*{Res}_{z=\frac{1}{3}} f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z=\frac{1}{3}} = \frac{30z^2 - 23z + 5}{(2z-1)^2 3 + (3z-1) \cdot 2(2z-1) \cdot 2}$$

$$= \frac{\frac{10}{3} - \frac{23}{3} + 5}{1/3 + 0} = \frac{2/3}{1/3} = \underline{\underline{2}}$$

$\therefore \displaystyle \operatorname*{Res}_{z=\frac{1}{3}} f(z) = 2$

By CRT, $\displaystyle \oint_C f(z)\, dz = 2\pi i \left[\operatorname*{Res}_{z=1/2} f(z) + \operatorname*{Res}_{z=\frac{1}{3}} f(z) \right]$

$$= 2\pi i \left[1/2 + 2 \right]$$

$$= \underline{\underline{5\pi i}}.$$

2) Evaluate $\int \frac{\tan z}{z^2-1}\, dz$ $\quad c;\ |z| = \frac{3}{2}$

<u>Soln:</u>

$z = \pm 1$ are singularities. simple poles

$(z^2-1) = (z+1)(z-1) \Rightarrow z=1$ and $z=-1$ are simple poles

$C;\ |z| = \frac{3}{2}$

at $z=1 \Rightarrow |1| = 1 < \frac{3}{2}$

at $z=-1 \Rightarrow |-1| = 1 < \frac{3}{2}$ $\Rightarrow$ both lies inside c.

~~Now~~ By CRT, $\oint_C f(z)\, dz = 2\pi i \left[\text{sum of Residues inside } C. \right]$

Now $\operatorname{Res}_{z=1} f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z=1}$

$$= \left. \frac{\tan z}{2z} \right|_{z=1} = \frac{\tan 1}{2}$$

and $\operatorname{Res}_{z=-1} f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z=-1}$

$$= \left. \frac{\tan z}{2z} \right|_{z=-1} = \frac{\tan(-1)}{-2}$$

$$= \frac{-\tan 1}{-2} = \frac{\tan 1}{2}$$

$\therefore$ By CRT, $\oint_C f(z)\, dz = 2\pi i \left[\frac{\tan 1}{2} + \frac{\tan 1}{2} \right]$

$$= 2\pi i \tan 1.$$

Q) Evaluate $\oint_C \frac{z^2}{(z-1)^2(z-2)}$ where C ; $|z| = 2.5$ using cauchy's Residue theorem.

Soln:

$$f(z) = \frac{z^2}{(z-1)^2(z-2)}$$

$|1| = 1 < 2.5$

$|2| = 2 < 2.5$

Singularities are :-

$z=1$ is a pole of order 2.

$z=2$ is a simple pole. $\Rightarrow$ both lies inside C.

By CRT, $\oint_C f(z)\,dz = 2\pi i \left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\, f(z) \right]$

Now $\underset{z=1}{\text{Res}}\, f(z)$ (pole of order 2 ; $m=2$)

$$= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

$$= \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^1}{dz^1} \left[(z-1)^2 \cdot \frac{z^2}{(z-1)^2 \cdot (z-2)} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{z^2}{z-2} \right]$$

$$= \lim_{z \to 1} \left[\frac{(z-2) \cdot 2z - z^2 \cdot 1}{(z-2)^2} \right]$$

$$= \frac{-1 \cdot 2 - 1}{1} = \frac{-3}{1} = \underline{\underline{-3}}$$

$\therefore \underset{z=1}{\text{Res}}\, f(z) = -3$ //

Now $\underset{z=2}{\text{Res}}\, f(z)$ = (Simple pole)

$$= \left.\frac{g(z)}{h'(z)}\right|_{z=2}$$

$g(z) = z^2$

$h(z) = (z-1)^2 \cdot (z-2)$

$$= \left.\frac{z^2}{(z-1)^2 \cdot 1 + (z-2)\, 2(z-1)\cdot 1}\right|_{z=2}$$

$$= \frac{4}{1+0} = \underline{\underline{4}}$$

$$\therefore \oint f(z)\, dz = 2\pi i \left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\, f(z) \right]$$

$$= 2\pi i \left[-3 + 4 \right]$$

$$= 2\pi i\, (1) = \underline{\underline{2\pi i}}$$

Q) Evaluate $\int \frac{z-3}{z^2+2z+5}\, dz$ where i) C is $|z| = 1$

ii) C ; $|z+1-i| = 2$

iii) C ; $|z+1+i| = 2$

using CRT.

Soln:

To find singularities, $z^2+2z+5 = 0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = \underline{-1 \pm 2i}$$

The singularities are $z = -1+2i$ and

$z = -1-2i$

i) C; $|z|=1$

when $z=-1+2i$, $|z| = |-1+2i| = \sqrt{5} > 1$, lies outside c

when $z=-1-2i$, $|z| = |-1-2i| = \sqrt{5} > 1$, lies outside c.

$\therefore$ By Cauchy's Integral Theorem,

$$\oint_C f(z)\, dz = 0$$

ii) C; $|z+1-i|=2$

when $z=-1+2i$, $|z| = |(-1+2i)+1-i| = |i| = \sqrt{1} = 1 < 2$, lies inside c.

when $z=-1-2i$, $|z| = |(-1-2i)+1-i| = |-3i| = \sqrt{9} = 3 > 2$, lies outside c.

Also, $z=-1+2i$ is a simple pole.

$$\therefore \operatorname{Res}_{z=-1+2i} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-1+2i}$$

$$= \left.\frac{z-3}{2z+2}\right|_{z=-1+2i}$$

where $g(z) = z-3$
$h(z) = z^2+2z+5$

$$= \frac{(-1+2i)-3}{2(-1+2i)+2}$$

$$= \frac{-4+2i}{4i}$$

$$\therefore \operatorname{Res}_{z=-1+2i} f(z) = \frac{-2+i}{2i}$$

$\therefore$ By Cauchy's Residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \left[\text{sum of residues at singular points within } C\right].$$

$$= 2\pi i \left[\operatorname*{Res}_{z=-1+2i} f(z) \right]$$

$$= 2\pi i \left[\frac{i-2}{2i} \right]$$

$$= \pi(i-2)$$

iii) C ; $|z+1+i| = 2$.

when $z=-1+2i$, $|z+1+i| = |(-1+2i)+1+i| = |3i| = 3 > 2$, lies outside C.

when $z=-1-2i$, $|z+1+i| = |(-1-2i)+1+i| = |-i| = 1 < 2$, lies inside C.

Also $z=-1-2i$ is a simple pole.

$$\therefore \operatorname*{Res}_{z=-1-2i} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-1-2i} \qquad \text{where } g(z) = z-3,\ h(z) = z^2+2z+5$$

$$= \left.\frac{z-3}{2z+2}\right|_{z=-1-2i}$$

$$= \frac{(-1-2i)-3}{2(-1-2i)+2} = \frac{-4-2i}{-4i} = \frac{2+i}{2i}$$

$\therefore$ By CRT, $\oint_C f(z)\,dz = 2\pi i \left[\text{Sum of residues at singularities within } C\right]$

$$= 2\pi i \left[\frac{2+i}{2i}\right] = \pi(2+i).$$

Q) Find all singular points and corresponding residues of $f(z) = \dfrac{z+2}{(z+1)^2(z-2)}$

Soln:

put $(z+1)^2(z-2) = 0$ $\therefore$ Singularities are, $z=-1$ is a pole of order 2.

$z=2$ is a simple pole.

Now Residue $f(z)$ at $z=-1$ $(m=2)$

$$\therefore \operatorname*{Res}_{z=-1} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m \cdot f(z)\right)$$

$$= \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d}{dz} \left[\frac{\cancel{(z-(-1))^2} \cdot (z+2)}{\cancel{(z+1)^2} \cdot (z-2)} \right]$$

$$= \lim_{z\to -1} \frac{d}{dz} \left[\frac{z+2}{z-2}\right]$$

$$= \lim_{z\to -1} \left[\frac{(z-2)\cdot 1 - (z+2)\cdot 1}{(z-2)^2}\right]$$

$$= \left[\frac{(-1-2) - (-1+2)}{(-1-2)^2}\right] = \frac{-3-(1)}{(-3)^2} = \frac{-4}{9}$$

Now $\operatorname*{Res}_{z=2} f(z)$ (simple pole)

$$= \left.\frac{g(z)}{h'(z)}\right|_{z=2}$$

$$= \left.\frac{z+2}{(z+1)^2\cdot 1 + (z-2)\cdot 2(z+1)}\right|_{z=2}$$

$$= \frac{4}{3^2+0} = \frac{4}{9}$$

$$\therefore \operatorname*{Res}_{z=2} f(z) = \frac{4}{9}$$

2) Find the residue of $\frac{e^z}{z^3}$ at its pole.

Soln:

$z=0$ is a pole of order 3. ie $m=3$.

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$= \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^{3-1}}{dz^{3-1}} \left[(z-0)^3 \cdot \frac{e^z}{z^3}\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} \left[e^z\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \left[e^z\right]$$

$$= \frac{1}{2!} e^0$$

$$= \frac{1}{2}$$

$$\frac{d}{dz}(e^z) = e^z$$

$$\frac{d^2}{dz^2} e^z = e^z$$

Q) Using Cauchy's Residue theorem evaluate

$$\oint_C \frac{\cosh \pi z}{z^2+4} \quad ; \quad C : |z| = 3$$

<u>Soln:</u>

Singularities are $z^2+4=0$

$$\Rightarrow (z+2i)(z-2i)=0$$

$$\Rightarrow z = \pm 2i ,$$

$\therefore$ Singular points are $z = +2i$ and $z = -2i$

both are simple poles.

when $z = 2i$; $|z| = |2i| = \sqrt{4} = 2 < 3$ lies inside C.

when $z = -2i$; $|z| = |-2i| = \sqrt{4} = 2 < 3$ lies inside C.

both points lies inside C.

Now $\operatorname{Res}_{z=2i} f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=2i} =$

$$= \left.\frac{\cosh \pi z}{2z}\right|_{z=2i}$$

$$= \frac{\cosh \pi(2i)}{2\cdot 2i} = \underline{\underline{\frac{\cosh 2\pi i}{4i}}} = \underline{\underline{\frac{1}{4i}}}$$

also $\operatorname{Res}_{z=-2i} f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=-2i}$

$$= \left.\frac{\cosh \pi z}{2z}\right|_{z=-2i}$$

$$= \frac{\cosh \pi(-2i)}{2\cdot -2i} = \frac{\cosh -2\pi i}{-4i} = \underline{\frac{1}{-4i}}$$

$\therefore$ By CRT, $\oint_C f(z)\,dz = 2\pi i \left[\underset{z=2i}{\text{Res}}\, f(z) + \underset{z=-2i}{\text{Res}}\, f(z) \right]$

$= 2\pi i \left[\frac{1}{4i} + \frac{1}{-4i} \right]$

$= \underline{\underline{0}}$

Q) Evaluate $\oint_C \frac{z-23}{z^2-4z-5}\,dz$; $C : |z-i| = 2$ using Residue Integration theorem.

Soln:

$z^2 - 4z - 5 = 0 \Rightarrow (z+1)(z-5) = 0.$

$\Rightarrow z = -1$, and $z = 5$ are singularities.

both are simple poles.

when $z = -1$; ~~$C: |z| = |-1| = 1 <$~~, $C: |z-i| = |-1-i| = \sqrt{2} < 2$; lies inside c.

when $z = 5$; $C: |z-i| = |5-i| = \sqrt{26} > 2$ lies outside c.

$\therefore$ By CRT, $\oint_C f(z)\,dz = 2\pi i$ [Sum of residues at singularity inside c].

$= 2\pi i \left[\underset{z=-1}{\text{Res}}\, f(z) \right]$

Now $\underset{z=-1}{\text{Res}}\, f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z=-1} = \left. \frac{z-23}{2z-4} \right|_{z=-1}$

$= \frac{-24}{-6} = \underline{\underline{4}}$

$\therefore$ By CRT $\oint_c f(z)\, dz = 2\pi i \left[4 \right]$

$= \underline{\underline{8\pi i}}$

APPLICATION OF RESIDUES TO EVALUATE REAL INTEGRALS

1. Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function on $\sin\theta$ and $\cos\theta$.

This integral can be reduced to complex line integral by means of the substitution

$z = e^{i\theta}$ ie $|z| = 1$.

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{z^2-1}{2iz}$$

The integral can be evaluated by Cauchy's Residue theorem.

Note:
$$\int_0^{\pi} f(\cos\theta, \sin\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta.$$

Q) Using contour integration evaluate

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta$$

Soln:

$$z = e^{i\theta} \quad ; \quad |z| = 1$$

$$d\theta = \frac{dz}{iz} \quad , \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta = \int_{C:|z|=1} \frac{1}{5+4\frac{(z^2+1)}{2z}} \frac{dz}{iz}$$

$$= \frac{1}{i} \int_C \frac{1}{\frac{4z^2+10z+4}{2\cancel{z}}} \cdot \frac{dz}{\cancel{z}}$$

$$= \frac{1}{i} \int_C \frac{1}{2z^2+5z+2}\, dz \quad \text{——— (1)}$$

Singular points are, $2z^2+5z+2=0$

$$\Rightarrow z = \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

$$\Rightarrow z = \frac{-1}{2} \text{ and } z = -2$$

singularities are $z = \frac{-1}{2}$ and $z = -2$.

$c ; |z| = 1$

When $z = \frac{-1}{2}$, $|z| = |\frac{-1}{2}| = \frac{1}{2} < 1$ lies inside c.

When $z = -2$, $|z| = |-2| = 2 > 1$ lies outside c.

Also $z = \frac{-1}{2}$ is a simple pole.

$$\therefore \operatorname*{Res}_{z=-\frac{1}{2}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-\frac{1}{2}}$$

$g(z) = 1$

$h(z) = 2z^2 + 5z + 2$

$$= \left.\frac{1}{4z+5}\right|_{z=-\frac{1}{2}}$$

$$= \frac{1}{3}$$

$\therefore$ By Cauchy's Residue theorem (CRT),

$$\oint_C f(z)\, dz = 2\pi i \left[\text{Sum of residues at singular points inside c.} \right].$$

$$= 2\pi i \left[\operatorname*{Res}_{z=-\frac{1}{2}} f(z) \right]$$

$$= 2\pi i \left[\frac{1}{3} \right] = \frac{2\pi i}{3}$$

$$\therefore (1) \Rightarrow \int_0^{2\pi} \frac{1}{5+4\cos\theta} = \frac{1}{i} \times \frac{2\pi i}{3}$$

$$= \frac{2\pi}{3}$$

Q) Evaluate $\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$ using contour integration.

Soln:

Let $z = e^{i\theta}$; $|z|=1$

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_{C;|z|=1} \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{1}{\frac{z^2+4z+1}{2\cancel{z}}} \frac{dz}{\cancel{z}}$$

$$= \frac{2}{i}\int_C \frac{1}{z^2+4z+1}\, dz \quad \text{— (1)}$$

To find singularities,

$$z^2+4z+1=0 \Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Singularities are $z = -2+\sqrt{3}$ and $z = -2-\sqrt{3}$.

When $z = -2+\sqrt{3}$; $|z| = |-2+\sqrt{3}| = \quad < 1$, lies inside c

when $z = -2-\sqrt{3}$; $|z| = |-2-\sqrt{3}| = |2-\sqrt{3}| > 1$ lies outside c.

and $z = -2+\sqrt{3}$ is simple pole.

Now
$$\operatorname*{Res}_{z=-2+\sqrt{3}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-2+\sqrt{3}}$$

$$= \left. \frac{1}{2z+4} \right|_{z=-2+\sqrt{3}} \qquad \text{where } g(z) = 1$$

$$h(z) = z^2+4z+1$$

$$= \frac{1}{-4+2\sqrt{3}+4} = \frac{1}{2\sqrt{3}}$$

$\therefore$ By CRT $\int_C f(z)\,dz = 2\pi i\left[\text{Sum of residues at singularity inside } C\right]$

$$= 2\pi i\left[\operatorname*{Res}_{z=-2+\sqrt{3}} f(z)\right]$$

$$= 2\pi i\left[\frac{1}{2\sqrt{3}}\right]$$

$$= \frac{\pi i}{\sqrt{3}}$$

$\therefore$ ① $\Rightarrow$ required integral,

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta = \frac{2}{i} \times \left[\frac{\pi i}{\sqrt{3}}\right]$$

$$= \frac{2\pi}{\sqrt{3}}$$

Q) Evaluate $\int_0^{\pi} \frac{1}{a+b\cos\theta}\, d\theta$, $a>b>0$ using contour integration.

Soln:

put $z = e^{i\theta}$, $C: |z|=1$

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta = \int_{C;|z|=1} \frac{1}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i} \int_C \frac{1}{\frac{bz^2+2az+b}{2\cancel{z}}} \frac{dz}{\cancel{z}}$$

$$= \frac{2}{i} \int_C \frac{1}{bz^2+2az+b}\, dz. \quad \text{--- (1)}$$

Now $bz^2+2az+b=0 \Rightarrow z = \frac{-2a \pm \sqrt{(2a)^2-4b^2}}{2b}$.

$$\Rightarrow z = \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b}$$

$$\Rightarrow z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$\therefore$ The singularities are,

$$z = \frac{-a+\sqrt{a^2-b^2}}{b} \quad \text{and} \quad z = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Here $z = \frac{-a + \sqrt{a^2 - b^2}}{b}$ lies inside C and

$z = \frac{-a - \sqrt{a^2 - b^2}}{b}$ lies outside C.

Also $z = \frac{-a + \sqrt{a^2 - b^2}}{b}$ is a simple pole.

$\therefore$
$$\operatorname*{Res}_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}} f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}}$$

$$= \left. \frac{1}{2bz + 2a} \right|_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}}$$

$$= \frac{1}{2b\left[\frac{-a + \sqrt{a^2 - b^2}}{b}\right] + 2a}$$

$$= \frac{1}{-2a + 2\sqrt{a^2 - b^2} + 2a} = \frac{1}{2\sqrt{a^2 - b^2}}$$

$\therefore$ By CRT, $\oint_C f(z)\, dz = 2\pi i \left[\text{sum of residues at singular points inside } C\right]$

$$= 2\pi i \left[\frac{1}{2\sqrt{a^2 - b^2}}\right] = \frac{\pi i}{\sqrt{a^2 - b^2}}$$

$\therefore$ ① $\Rightarrow$ $\oint_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{2}{i} \times \frac{\pi i}{\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ //.

2. 2. Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$

where $f(x)$ and $g(x)$ are polynomials such that no ~~zeros~~ poles ~~on~~ lies on the real axis.

Then $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx = \int_C \frac{f(z)}{g(z)} dz$ which

can be evaluated by using Residue theorem where C is the upper semi circle.

also $\int_{0}^{\infty} \frac{f(x)}{g(x)} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$

Q) Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx$ using contour integration.

Soln:

~~g(z)~~ Integrals is of the form $\int \frac{f(x)}{g(x)} = \int \frac{f(z)}{g(z)}$

~~where $g(z) = z^4+10x^2$~~

$g(x) = x^4+10x^2+9$

$\therefore g(z) = z^4+10z^2+9$

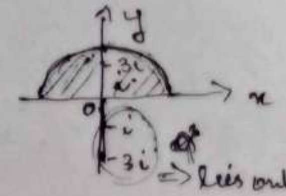
put $g(z) = z^4+10z^2+9 = 0$

$\Rightarrow (z^2)^2+10(z^2)+9 = 0$

$\Rightarrow z^2 = \frac{-10 \pm \sqrt{100-36}}{2} = \frac{-10\pm 8}{2} = -1, -9$

$$\Rightarrow z^2 = -1 \text{ and } z^2 = -9$$

$$\Rightarrow z = \pm i \text{ and } z = \pm 3i$$



Here $z = -i, -3i$ lies outside the upper semi circle c.

$z = i, +3i$ lies inside upper semi circle.

also $z = i$ and $z = 3i$ are simple poles.

$\therefore$ By Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\,dx = \int_C \frac{f(z)}{g(z)}\,dz$$

$$= 2\pi i \left[\text{Sum of residues at singular points within } c.\right].$$

Now
$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=i}$$

$$= \left.\frac{z^2 - z + 2}{4z^3 + 20z}\right|_{z=i}$$

where $g(z) = z^2 - z + 2$

$h(z) = z^4 + 10z^2 + 9$

$$= \frac{-1 - i + 2}{-4i + 20i} = \frac{1 - i}{16i}$$

Now
$$\operatorname*{Res}_{z=3i} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=3i}$$

$$= \left.\frac{z^2 - z + 2}{4z^3 + 20z}\right|_{z=3i} = \frac{-9 - 3i + 2}{-108i + 60i}$$

$$= \frac{-3i-7}{-48i} = \frac{3i+7}{48i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\,dx = 2\pi i \left[\frac{1-i}{16i} + \frac{3i+7}{48i} \right]$$

$$= 2\pi i \left[\frac{10}{48i} \right] = \frac{5\pi}{12}$$

Q) Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

Soln:

Let $g(z) = z^4+1$

$g(z)=0 \Rightarrow z^4+1=0 \Rightarrow z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

Here $z = e^{i5\pi/4}$ and $z = e^{i7\pi/4}$ lies outside the uppersemi circle C. $\left[\because\right.$ 3rd quadrant & 4th quadr

~~Bec~~ And $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$ lies inside C.

Both are simple poles.

Now $\underset{z=e^{i\pi/4}}{\text{Res}}\, f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=e^{i\pi/4}}$ $\left[\because\ g(z)=1,\ h(z) = z^4+1\right.$

$$= \left.\frac{1}{4z^3}\right|_{z=e^{i\pi/4}}$$

$$= \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4} e^{-3i\pi/4}$$

$$= \frac{1}{4}\left(\cos 3\pi/4 - i \sin 3\pi/4\right)$$

$$= \frac{1}{4}\left(\frac{-1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right)$$

Now $\underset{z=e^{i3\pi/4}}{\text{Res}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=e^{i3\pi/4}} = \left.\frac{1}{4z^3}\right|_{z=e^{i3\pi/4}}$

$$= \frac{1}{4\cdot\left(e^{i3\pi/4}\right)^3} = \frac{1}{4} e^{-i9\pi/4}$$

$$= \frac{1}{4}\left[\cos(9\pi/4) - i \sin(9\pi/4)\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right]$$

$\therefore$ By CRT, $\int \frac{f(z)}{g(z)} dz = 2\pi i \left[\begin{array}{l}\text{Sum of residues at} \\ \text{singular point inside } c\end{array}\right]$

$$= 2\pi i \left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\pi i \left[\cancel{-\frac{1}{4}\cdot\frac{1}{\sqrt{2}}} - i\frac{1}{4}\cdot\frac{1}{\sqrt{2}} + \cancel{\frac{1}{4}\cdot\frac{1}{\sqrt{2}}} - i\frac{1}{4}\cdot\frac{1}{\sqrt{2}}\right]$$

$$= 2\pi i \left[\frac{-2i}{4\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx = \frac{\pi}{\sqrt{2}}$$

Q) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$

Soln:

The singularities are $g(z) = 0$

$$\Rightarrow (z^2+a^2)(z^2+b^2) = 0$$

$$\Rightarrow z^2 = -a^2 \text{ and } z^2 = -b^2$$

$$\Rightarrow z = \pm ia \text{ and } z = \pm ib.$$

The singular points $z = -ia$ and $z = -ib$ lies outside the upper semicircle C.

The singular points $z = ia$ and $z = ib$ lies inside the upper semi circle C.

Now $\underset{z=ia}{\text{Res}}\, f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=ia} = \left.\dfrac{z^2}{(z^2+a^2)\cdot 2z + (z^2+b^2)\cdot 2z}\right|_{z=ia}$

$$= \frac{-a^2}{0 + (-a^2+b^2)\cdot 2ia} = \frac{-a}{2i(-a^2+b^2)}$$

$$= \frac{-a}{-2i(a^2-b^2)} = \underline{\underline{\frac{a}{2i(a^2-b^2)}}}$$

$\underset{z=ib}{\text{Res}}\, f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=ib} = \left.\dfrac{z^2}{(z^2+a^2)\cdot 2z + (z^2+b^2)\cdot 2z}\right|_{z=ib}$

$$= \frac{-b^2}{(-b^2+a^2)2ib + 0} = \frac{-b}{2i(-b^2+a^2)} = \frac{-b}{2i(a^2-b^2)}$$

$$= \frac{-b}{2i(a^2-b^2)} \;//$$

$\therefore$ By CRT, $\int_C \frac{f(z)}{g(z)}\,dz = 2\pi i$ [sum of residues at singular point inside C]

$$= 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)}\right]$$

$$= 2\pi i \left[\frac{(a-b)}{2i(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{(a-b)}{(a+b)(a-b)}\right]$$

$$= \pi\left(\frac{1}{a+b}\right)$$

$$= \frac{\pi}{a+b}$$